0-1 Laws: The Σ_1^1 Bernays-Schönfinkel and Σ_1^1 Ackermann Classes

Michael Sands

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Introduction

In the past two decades, the subject of finite model theory has attracted increasing attention from logicians, mathematicians, and computer scientists. Though closely related to standard model theory, the subject requires a different methodology, e.g. a tendency to use probability arguments to prove that certain properties hold for "almost all" finite models, and game-theoretic arguments to find finite models that witness differences in expressive power of various logics. This essay is devoted to one particular corner of finite model theory, 0-1 laws, which concern the probability that a given finite structure models a certain sentence in a language.

We define C as the class of finite relational structures over some vocabulary, $\mathbf{R}$, such that their universes are initial segments of the integers, i.e. $\{1, 2, \ldots, m\}$. Because we have restricted ourselves to structures with these types of universes, we will deal only with labeled 0-1 laws rather than unlabeled laws (for a sketch of unlabeled laws and their proofs, see [5]). Let F be a formula, expressible in some logic L.

Definition 1. *The **model probability** of F, $\mu_m(F)$, is the fraction of structures with m elements that satisfy F. That is:*

$$\mu_m(F) = \frac{\textit{Number of structures of size } m \textit{ that satisfy } F}{\textit{Number of all structures of size } m}$$

Definition 2. *We call $\lim\limits_{m\to\infty} \mu_m(F)$ the **asymptotic probability**, and denote it $\mu(F)$. If $\mu(F) = 1$, we say that F is* almost surely true *on C. If $\mu(F) = 0$, we say that F is* almost surely false *on C.*

For a logic L, if $\mu(F)$ exists and is equal to 0 or 1 for every formula F expressible in logic L, we say that L obeys a 0-1 law.

Glebsky proved that a 0-1 law holds for first-order logic in 1969 [9], but it was proved independently in the Western world by Fagin in 1972 [8]. The question naturally turned to second-order logic. However, it was clear that a 0-1 Law could not hold for the entirety of second-order logic, since one can formulate the statement, 'the universe has an even number of elements' by defining a one-to-one relation that correlates elements of the universe to other distinct elements in the universe. In fact, we can do this with a single existential second-order quantifier. The asymptotic probability of this sentence will not converge because it will oscillate between 0 and 1. Thus, the law does not hold.

The inquiry into the 0-1 laws thus turned to certain *fragments* of second-order logic, and whether these fragments obey a 0-1 law. We will look at two different fragments of Σ_1^1 sentences, which are defined by their first-order quantifier prefixes. Consider the following three classes of first-order sentences, where φ is a formula with no quantifiers:

- **Bernays-Schönfinkel class:** $\exists \cdots \exists \forall \cdots \forall (\varphi)$
- **Ackermann class:** $\exists \cdots \exists \forall \exists \cdots \exists (\varphi)$
- **Gödel class:** $\exists \cdots \exists \forall \forall \exists \cdots \exists (\varphi)$

The first few results of finite model theory regarding 0-1 laws suggested a parallel between decidability for the first-order class of sentences and 0-1 laws for the Σ_1^1 fragments defined by these prefixes. Both the Bernays-Schönfinkel and Ackermann first-order classes were proved to be decidable in the 1920s. By the 1980s, proofs existed for 0-1 laws of the Σ_1^1 Bernays-Schönfinkel and Σ_1^1 Ackermann classes. Moreover, it was proven that the Σ_1^1 fragments that correspond to the minimal undecidable first-order classes ($\forall\forall\forall\exists$, $\forall\exists\forall$, and $\forall\forall\exists$ with equality) did not obey 0-1 laws. The hope for a deep connection between decidability of first-order classes and 0-1 laws for the corresponding Σ_1^1 fragments disappeared when the Gödel class without equality was proven not to obey a 0-1 law [1], even though it is decidable [10].

In this paper, I will devote a significant amount of attention to the Bernays-Schönfinkel and Ackermann classes. In §2, I will give an overview of random graphs. This background will serve as a foundation for a discussion of random structures. Of particular importance is the **countable random structure**, which is essential to the Phokion Kolaitis and Moshe Vardi proof of the 0-1 laws for the Bernays-Schönfinkel and Ackermann classes [18]. In §3, I will discuss the standard way of proving a 0-1 law: Transfer Theorems. Then, in §4, I will present a (nearly trivial) proof for the decidability of the Bernays-Schönfinkel first-order class, followed by a proof of a 0-1 law for the Σ_1^1 Bernays-Schönfinkel class. In this treatment, I will hew closely to the aforementioned Kolaitis and Vardi proof.

In §5, I will give a decidability proof for the Ackermann class; in §6, a proof of the finite controllability of the Ackermann Class; and finally, in §7, a proof for the 0-1 law for the Σ_1^1 Ackermann class. After presenting the final proof, a refinement of Kolaitis and Vardi, I shall reflect on the advantages of my proof. Finally, I will briefly remark on Le Bars' proof of the failure of the 0-1 law of the Gödel class (without equality), and present Le Bars' conjecture that revisits the hope of finding a connection between decidability and 0-1 laws.

1 Basic Definitions and Notation

In this section, we will lay the groundwork for our model theory. It will necessarily be brief and avoid much of the fine detail, though I will define all concepts used later.

Our language will be comprised of

- connectives: '$\wedge$' (and), '$\vee$' (or), '$\neg$' (not), and '$\rightarrow$' (if...then)
- variables (x_0, $x_1, \ldots, x_n, \ldots$),
- parentheses '(' and ')',

- quantifiers '$\forall$' (for all) and '$\exists$' (exists),

- a binary relation '$=$' (identity).

We could restrict the set of connectives to $\wedge$ and $\neg$, as these are expressively adequate, but such parsimony offers no advantages for our analysis.

We will use Greek lowercase letters φ, ψ, etc. to denote sentences of the language, lowercase Roman letters from earlier in the alphabet a, b, c, d, etc. to denote elements of the model (or instantiated variables), and lowercase Roman letters from later in the alphabet, $x_i, v_i,$, etc. to denote variables.

Definition 3. *A **relational vocabulary** is a finite set that consists of predicate letters $\mathcal{P}, \mathcal{Q}, \mathcal{R}, \ldots \mathcal{S}$. We use* $\mathbf{R}$ *and* $\mathbf{S}$ *to denote **vocabularies**.*

A vocabulary is traditionally defined as above, but with the addition of constants and function signs in addition to predicate letters (also known as 'relational symbols'). However, our subject matter prohibits constants in any of the classes of formulas we treat. For if c is some constant in the language, with $\mathcal{R}$ a unary predicate letter in the vocabulary, then $\mu(\mathcal{R}c) = \frac{1}{2}$ [13]. This can be seen easily by pairing every structure that makes $\mathcal{R}c$ true with another structure that makes $\mathcal{R}c$ false. Such a pairing can be demonstrated by keeping the interpretation of c the same as in the first structure, but re-interpreting $\mathcal{R}$ in the second structure to be the compliment of $\mathcal{R}$ in the first structure.

Definition 4 ($\mathbf{R}$-structure)**.** *A **model*** $\mathbf{M}$ *of a vocabulary* $\mathbf{R}$ *is:*

1. *A non-empty set, $U_{\mathbf{M}}$, known as the universe of* $\mathbf{M}$*, and*

2. *for every k-ary relational symbol in the language* $\mathbf{R}$*, a k-ary relation in the universe.*

Note: For clarity, we will omit the use of $U_{\mathbf{M}}$ and say '$a \in \mathbf{M}$' or 'a is an element of $\mathbf{M}$', whenever $a \in U_{\mathbf{M}}$. Such ambiguity of usage will not affect our discussion, as the context will always be clear.

Definition 5 (Models)**.** *For a sentence ψ in the language L, we say that* $\mathbf{M}$ ***models*** *ψ, when ψ is true in* $\mathbf{M}$*. We denote this '$\mathbf{M} \models \psi$'.*

These two terms, 'model' and 'models', can create some confusion, as there are two possible meanings for '$\mathbf{M}$ is a model'. The first possibility adheres more to definition 4. In this case, the sentence merely asserts that $\mathbf{M}$ is a structure with elements and relations. The second possible meaning is derivative of definition 5. Under this meaning, the sentence asserts that $\mathbf{M}$ models some sentence in the language. I have differentiated them here by making the first a noun and the second a verb, though this is not always such a clear distinction. I will use 'structure' for the former meaning and 'model' for the latter.

2 The Random Graph and the Random Structure

The countable random structure is essential to the proof of the 0-1 laws as given by Kolaitis and Vardi [18]. They employ the countable random structure to show that a Transfer Theorem holds, which was shown by Fagin [8] to be equivalent to proving that a 0-1 law holds.

We will lay the foundation for the countable random structure by discussing random graphs. This will involve some graph theory, but the specific details of graph theory are not our primary concern. Instead, we will refine our intuitions about randomness and observe the general strategy of *back-and-forth* arguments in building an isomorphism.

> Imagine the following infinite experiment: For each pair $\{i, j\}$ of distinct positive integers, toss a fair coin to decide if $\langle i, j\rangle$ is an edge. Outcomes

> of the experiment are graphs on positive integers. What do you think is the probability that two outcomes are isomorphic? [13]

The intuition is to say that two such graphs, determined by coin flips, could never be isomorphic to each other. Our pre-reflective intuition of randomness tells us that things randomly created in this way cannot be alike. If two such graphs were isomorphic, it would imply that every graph created in such a manner is isomorphic to every other such graph, and we would have an equivalence class of graphs.

However, this initial intuition is incorrect. We can demonstrate the isomorphism by building it piecemeal. Suppose we have a partial isomorphism f with finite domain, from one outcome graph to the other. We will prove that we can extend f to f'—a partial isomorphism with domain one element larger than f—by way of a probability argument. Since f is finite, let $i \in \mathbb{N}$ be such that $i \notin \mathrm{dom}(f)$. To extend the isomorphism to include i in the domain, take $j \notin \mathrm{ran}(f)$. The probability that another partial isomorphism results from extending f to f' with $f'(i) = j$ is $\left(\frac{1}{2}\right)^k$, with $k = |\mathrm{dom}(f)|$. This probability results from the multiplication of the probabilities that $f'(i)$ stands in the proper relations to each of the images of the elements in the domain of f. If we select n-many such j's, the probability that at least one of the selected j's is appropriate is $1 - \left(1 - \left(\frac{1}{2}\right)^k\right)^n$. This quantity tends to 1 as n grows. Thus, the probability of being able to extend f to f' to include $i \in \mathrm{dom}(f')$ is 1.

To build the isomorphism between the two randomly generated graphs, we start with an empty partial isomorphism and extend it to the smallest integer not in the domain. The probability that we can do so is 1. Next, take the smallest integer not in the range. By the same probability argument, we can extend f again to include this integer in the range with probability 1. Then, since the intersection of a countable number of events with probability 1 has probability 1, we can extend the empty partial isomorphism to a complete isomorphism using this back-and-forth technique.

We can adapt our construction of a random graph to a random construction of a finite structure A (the following example is taken from [7]). Let $\mathbf{K}$ be some class of structures for the relational vocabulary $\mathbf{R}$. We repeat the same experiment as above, but this time for all predicates in $\mathbf{R}$, rather than for integers $\langle i, j\rangle$. Let $\{1, \ldots, m\}$ be the finite universe for the structure A. We will randomly determine the truth values for all n-ary predicates $\mathcal{R} \in \mathbf{R}$. For every possible selection of $\langle a_1, \ldots, a_n\rangle \in \{1, \ldots, m\}$, for all $1 \leq n \leq m$, we toss a fair coin to determine if $\mathcal{R}\, a_1, \ldots, a_n$ is true. Then, $\mu_m(\mathbf{K})$ is the probability that A, the randomly created structure of size m, belongs to $\mathbf{K}$.

Rather than restrict ourselves to m many elements in the model, we will do this an infinite number of times and call the result $\mathbf{A}$, the **countable random structure** over vocabulary $\mathbf{R}$ (we shall later show that $\mathbf{A}$ is unique up to isomorphism). That is, we define $\mathbf{A}$ to be the result of iterating the above process for a structure with universe equal to $\{1, 2, \ldots, n, \ldots\}$.

Definition 6 (n-types)**.** *An* ***n-type*** *for a vocabulary* $\mathbf{R}$ *is a maximally consistent set of atomic formulas and negations of atomic formulas constructed from predicate letters of* $\mathbf{R}$ *and variables among* $x_1, \ldots, x_n$.

In contrast to the usage in ordinary model theory, our n-types are quantifier-free, and hence for any structure $\mathbf{M}$ and a fixed n, the number of possible n-types is finite. In particular, the number of 1-types possible in an $\mathbf{R}$-model is 2^k, where k is the number of predicate letters in $\mathbf{R}$.

Definition 7 (Extension)**.** *Given* $\mathbf{t}$*, an n-type, we say* $\mathbf{t}'$ ***extends*** $\mathbf{t}$ *if* $\mathbf{t}'$ *is an* m*-type with* $m > n$*, and* $\mathbf{t} \subset \mathbf{t}'$*.*

Definition 8 (Realization)**.** *The n-type* ***realized*** *by* $\langle a_1, \ldots, a_n\rangle$ *in* $\mathbf{M}$ *is the unique n-type* $\mathbf{t}$ *such that for all* $\varphi \in \mathbf{t}$*,* $\mathbf{M} \models \varphi[a_1, \ldots, a_n]$

The type $\mathbf{t}$ realized by $\langle a_1, \ldots, a_n\rangle$ is denoted $\mathbf{t} = \mathbf{tp}_{\mathbf{M}}(a_1, \ldots, a_n)$.

Definition 9 (Restriction). *If* $\mathbf{t}$ *is an* n*-type, then* $\mathbf{t}$ ***restricted*** *to a variable* x_i *is the unique 1-type* $\mathbf{s}$ *such that for all* φ*,* $\varphi \in \mathbf{s} \leftrightarrow \varphi(x_1/x_i) \in \mathbf{t}$*.*

We denote the **restriction** of $\mathbf{t}$, an n-type, to the variable x_i by $[\mathbf{t}|\, x_i]$.

Definition 10 (Extension Axiom). *Given an* n*-type* $\mathbf{t}$ *and any* $(n+1)$*-type* $\mathbf{t}'$ *that extends* $\mathbf{t}$*, there is a first order extension axiom*

$$\tau : \forall x_1 \cdots \forall x_n(\mathbf{t} \rightarrow (\exists x_{n+1})\mathbf{t}')$$

Note that for $n = 0$, the extension axiom is simply $\tau :\ (\exists x_1)\mathbf{t}$, for any 1-type $\mathbf{t}$. Now let T be the set of all extension axioms. The countable random structure $\mathbf{A}$ is a countable model for T. Using a back-and-forth argument similar to the one we used earlier for random graphs, we can build in a stepwise fashion an isomorphism between any two countable models of T. Thus, the countable random structure is unique, up to isomorphism.

Here it will be instructive to use this property of $\mathbf{A}$ to prove explicitly that any finite model can be isomorphically embedded into $\mathbf{A}$, the countable random structure. The following proof will be an argument similar to the above back-and-forth argument, except it will only involve the "forth" direction.

Theorem 1 (Finite Embedding). *Every finite model* $\mathbf{M}$ *is isomorphically embedded in* $\mathbf{A}$*.*

Proof. Let $\mathbf{M}$ be a finite model, with elements $b_1, \ldots, b_m$ in the universe. We will construct an isomorphism $f : \mathbf{M} \longrightarrow \mathbf{A}$ by induction. In building this isomorphism, it will be sufficient to prove that the types are isomorphic, since this will imply the preservation of the relations in addition to the elements of the respective models.

Base Case: For $b_1 \in \mathbf{M}$, there is an extension axiom $\tau \in T$ such that we can extend the empty-type to a 1-type in any way. This is due to the randomness property of $\mathbf{A}$, which "says" that anything that can happen in the structure, does. In particular, there is some $a_1 \in \mathbf{A}$ such that $\mathbf{tp_A}(a_1) = \mathbf{tp_M}(b_1)$. Set $f(b_1) = a_1$.

Inductive Step: Suppose we have a partial isomorphism f for $b_1, \ldots, b_n$ with $n < m$, such that $f(b_i) = a_i$, for $1 \leq i \leq n$. Let

$$\mathbf{t} = \mathbf{tp_M}(b_1, \ldots, b_n) = \mathbf{tp_A}(a_1, \ldots, a_n)$$

Letting $\mathbf{t}' = \mathbf{tp_M}(b_1, \ldots, b_{n+1})$, there is some $\tau \in T$, an extension axiom such that

$$\tau = \forall x_1 \cdots \forall x_n \left(\mathbf{t} \to (\exists x_{n+1})\mathbf{t}'\right)$$

This is essentially an instance of a more general property of $\mathbf{A}$: in the model, anything that could possibly happen, does happen. In this case, for any possible way in which a type could be extended in $\mathbf{A}$, there is an extension of the type in just that way. Since we can extend the type, there is some $a_{n+1} \in \mathbf{A}$ such that

$$\mathbf{tp_M}(b_1, \ldots, b_n, b_{n+1}) = \mathbf{tp_A}(a_1, \ldots, a_n, a_{n+1})$$

To extend the isomorphism, set $f(b_{n+1}) = a_{n+1}$. Thus, we can build an isomorphism for any size m, and $\mathbf{M}$ is isomorphic to some sub-structure of $\mathbf{A}$.

$\square$

There is an immediate corollary, if we let $m \to \infty$:

Corollary 1. *Any countable structure* $\mathbf{B}$ *over vocabulary* $\mathbf{R}$ *is isomorphically embedded in* $\mathbf{A}$.

Recall that C is defined as the class of all finite structures with universe $\{1, \ldots, m\}$ over the relational vocabulary $\mathbf{R}$. We want to show that each of the extension axioms is almost surely true on C.

Theorem 2. *On the class C, for any extension axiom τ, τ is almost surely true.*

Proof. For a given $a_1, \ldots, a_n, c \in \mathbf{M}$, a structure in C, we have:

$$\text{Prob}\left[\mathbf{t}' \neq \mathbf{tp_M}(a_1, \ldots, a_n, c)\right] \leq \left(1 - \frac{1}{q}\right)$$

where q is the number of all possible $(n+1)$-types. Remember that q is fixed and finite for a given n. Then,

$$\text{Prob}\left[\mathbf{t} = \mathbf{tp_M}(a_1, \ldots, a_n)\ \&\ \mathbf{t}' \neq \mathbf{tp_M}(a_1, \ldots, a_n, c)\right] \leq \left(1 - \frac{1}{q}\right)$$

Now, because the universe of $\mathbf{M} = \{1, \ldots, m\}$, for a given $\langle a_1, \ldots, a_n \rangle$

$$\text{Prob}\left[(\nexists c)\Big(\mathbf{t} = \mathbf{tp_m}(a_1, \ldots, a_n)\ \&\ \mathbf{t}' = \mathbf{tp_M}(a_1, \ldots, a_n, c)\Big)\right] \leq \left(1 - \frac{1}{q}\right)^{m-n}$$

Since there are m^n different ways of choosing $\langle a_1, \ldots, a_n \rangle$,

$$\text{Prob}\left[(\exists a_1 \cdots \exists a_n)(\nexists c)\Big(\mathbf{t} = \mathbf{tp_m}(a_1, \ldots, a_n)\ \&\ \mathbf{t}' = \mathbf{tp_M}(a_1, \ldots, a_n, c)\Big)\right] \leq m^n \left(1 - \frac{1}{q}\right)^{m-n}$$

However, $m^n \left(1 - \frac{1}{q}\right)^{m-n}$ tends to 0 as m tends to ∞. Thus, $\mu(\tau) = 1$. □

This theorem will later be useful in §4, in our proof of a 0-1 law for the Bernays-Schönfinkel Σ_1^1 class.

3 Transfer Theorems

The standard way to prove a 0-1 law for a logic, attributed to Fagin [8] and used by Kolaitis and Vardi [18], is to show that a Transfer Theorem holds for the logic. Essentially, a Transfer Theorem says that for every sentence in the logic, $\mathbf{A}$ models the sentence if and only if the asymptotic probability of the sentence is 1. We will state a Transfer Theorem for first-order logic, and then prove that it holds via a compactness argument.

Theorem 3 (**Transfer Theorem: First-Order Logic**). *Let φ be any first-order formula with relational vocabulary $\mathbf{R}$. Let $\mathbf{A}$ be the countable random structure for $\mathbf{R}$. Then, if $\mathbf{A} \models \varphi$ then $\mu(\varphi) = 1$.*

Proof. Let T be the set of all extension axioms that characterize $\mathbf{A}$. Because any two countable models of T are isomorphic, and T has only infinite models, we know that T is syntactically complete (Vaught's Test). Then, since T is syntactically complete, $T \models \varphi$. By compactness of first-order logic (remember that T is a set of first-order sentences), we have some finite subset T_0 of T that implies φ, or $T_0 \models \varphi$. However, since T_0 is a finite collection of extension axioms,

$$\mu\left(T_0 \text{ is true in a model of size } m\right) = 1$$

Thus, $\mu(\varphi) = 1$. □

Corollary 2. *First-order logic obeys a 0-1 law.*

Proof. Let φ be a first-order sentence.

If $\mathbf{A} \models \varphi$, then $\mu(\varphi) = 1$ by Theorem 3.

If $\mathbf{A} \models (\neg\varphi)$, then $\mu(\neg\varphi) = 1$, and thus, $\mu(\varphi) = 0$.

Under the structure $\mathbf{A}$, either $\mathbf{A} \models \varphi$ or $\mathbf{A} \models (\neg\varphi)$.

□

We have proven that a Transfer Theorem holds for first-order logic, and therefore that a 0-1 law holds. We want to try and extend this theorem to second-order logic, but a Transfer Theorem cannot hold for arbitrary Σ_1^1 sentences, since we can formulate a Σ_1^1 sentence, φ, that says, 'There is a permutation on the universe where each element has order 2'. In finite models, φ is equivalent to saying 'The universe has an even number of elements'. We can see that the probability $\mu(\varphi)$ will oscillate between 0 and 1 and will not converge. Yet, φ is true on $\mathbf{A}$, the countable random structure. A witness is the permutation that interchanges every odd number j with $j+1$. This permutation is composed entirely of elements with order 2 and is well defined for all $\mathbb{N}$. Another counterexample is the statement 'there is a complete ordering with no maximum element.' This statement, also expressible as a Σ_1^1 sentence, is clearly true on $\mathbf{A}$, but fails for every finite model.

However, we can prove a Transfer Theorem for the class of Π_1^1 sentences, and, conversely, the negative direction for the class of Σ_1^1 sentences.

Theorem 4. *Let $\boldsymbol{A}$ be the countable random structure over $\boldsymbol{R}$, as above, and let $(\forall \mathbf{S})\Theta(\mathbf{S})$ be an arbitrary Π_1^1 sentence. If $\boldsymbol{A} \models (\forall \mathbf{S})\Theta(\mathbf{S})$, then there is a first-order sentence ψ over vocabulary $\mathbf{R}$ such that $\mu(\psi) = 1$ and $\models \psi \rightarrow (\forall \mathbf{S})\Theta(\mathbf{S})$*

Thus, for every Π_1^1 sentence φ, if $\mathbf{A} \models \varphi$, then $\mu(\varphi) = 1$.

Proof. We want to show the existence of the first-order sentence ψ as above, so assume $\mathbf{A} \models (\forall \mathbf{S})\Theta(\mathbf{S})$. Moreover, in order to obtain a contradiction, assume $T \cup \neg\Theta(\mathbf{S})$ is satisfiable. Let $\mathbf{B}$ be a structure such that $\mathbf{B} \models T \cup \neg\Theta(\mathbf{S})$. Now we define $\mathbf{B}'$ to be $\mathbf{B}$ reduced to $\mathbf{R}$. That is, $\mathbf{B}'$ is $\mathbf{B}$ with all $\mathbf{S}$-relations thrown out, retaining all of and only the $\mathbf{R}$-relations. Note that $\mathbf{B}' \cong \mathbf{A}$ because $\mathbf{B}'$ is a model for T. But then, $\mathbf{A} \models (\exists \mathbf{S})\neg\Theta(\mathbf{S})$, which is a contradiction.

Thus, $T \cup \neg\Theta(\mathbf{S})$ is not satisfiable. Since T is composed entirely of first-order τ's, by compactness there is some finite subset T' such that $T' \cup \neg\Theta(\mathbf{S})$ is not satisfiable.

Therefore, $(\bigwedge T') \longrightarrow \Theta(\mathbf{S})$ is valid. Hence, $\models (\bigwedge T') \longrightarrow (\forall \mathbf{S})\Theta(\mathbf{S})$. $(\bigwedge T')$ is our ψ.

□

We have an immediate corollary to this theorem:

Corollary 3 (Negative Σ_1^1 Transfer Theorem)**.** *Every Σ_1^1 sentence that is false on $\mathbf{A}$ has probability 0 on C.*

Theorem 4 shows that every Π_1^1 sentence satisfies the forward direction of a Transfer Theorem. Corollary 3 establishes the negative direction of a Transfer Theorem for Σ_1^1 sentences through the contrapositive. In order to show that a 0-1 law holds for the two fragments we will consider, it will suffice to show that if Φ is a sentence true on $\mathbf{A}$, then $\mu(\Phi) = 1$. We will prove this direction for two fragments of second-order logic: the Σ_1^1 Bernays-Schönfinkel and the Σ_1^1 Ackermann classes.

4 0-1 Law for the Σ_1^1 Bernays-Schönfinkel Class

Before proving the 0-1 law for the Σ_1^1 Bernays-Schönfinkel class, we will proceed through a proof that the first-order Bernays-Schönfinkel class is finitely controllable, and hence decidable.

Theorem 5 (Bernays-Schönfinkel Finite Controllability)**.** *Let*

$$\Phi = \exists x_1 \cdots \exists x_n \forall y_1 \cdots \forall y_m \varphi(x_1, \ldots, x_n, y_1, \ldots, y_m)$$

with φ lacking quantifiers. If $\mathbf{M}$ is a model for Φ, there is a finite model $\mathbf{M}'$ for Φ. In fact, $\mathbf{M}'$ is a submodel of $\mathbf{M}$, with cardinality n.

Proof. Let $\mathbf{M} \models \Phi$. Let $a_1, \ldots, a_n$ be elements of $\mathbf{M}$ that witness the existential quantifiers. Now let $\mathbf{M}'$ be $\mathbf{M}$ with its universe limited to $a_1, \ldots, a_n$. Since universal statements remain true under restrictions, $\mathbf{M'} \models \forall y_1 \cdots \forall y_m \varphi(a_1, \ldots, a_n, y_1, \ldots, y_m)$. Hence, by existential generalization, $\mathbf{M'} \models \Phi$. $\square$

Theorem 6 (0-1 Law for the Σ_1^1 Bernays-Schönfinkel Class)**.** *Let*

$$\Theta = (\exists \mathbf{S})(\exists x_1 \cdots \exists x_m)(\forall y_1 \cdots \forall y_n)\theta(x_1, \ldots, x_m, y_1, \ldots, y_n, \mathbf{R}, \mathbf{S})$$

be a Σ_1^1 Bernays-Schönfinkel sentence true on the countable random structure $\boldsymbol{A}$. Then, there is a first-order sentence ψ over $\mathbf{R}$ such that $\mu(\psi) = 1$, and

$$\psi \rightarrow (\exists \mathbf{S})(\exists x_1 \cdots \exists x_m)(\forall y_1 \cdots \forall y_n)\theta(x_1, \ldots, x_m, y_1, \ldots, y_n, \mathbf{R}, \mathbf{S})$$

is true in all finite structures. Thus, if Θ is a Σ_1^1 Bernays-Schönfinkel sentence true on $\boldsymbol{A}$, then $\mu(\Theta) = 1$.

Proof. Let $\mathbf{S}^*$ be an interpretation of $\mathbf{S}$-predicates such that by adjoining them to $\mathbf{A}$, we get a structure $\mathbf{A}^*$ such that

$$\mathbf{A}^* \models (\exists x_1 \cdots \exists x_m)(\forall y_1 \cdots \forall y_n)\theta(x_1, \ldots, x_m, y_1, \ldots, y_n, \mathbf{R}, \mathbf{S})$$

Now let $a_1, \ldots, a_m \in \mathbf{A}^*$ be such that they witness the first-order existentially quantified variables $x_1, \ldots, x_m$. Let $\mathbf{A_0} = [\mathbf{A}^* | (a_1, \ldots, a_m)]$. We define ψ, a first-order sentence that is a conjunction of a finite number of extension axioms. Specifically, if we let $\mathbf{tp_A}(a_1, \ldots, a_i) = \mathbf{t_i}$ for all $i \in \{1, \ldots, m\}$, then

$$\psi = (\exists x_1)\mathbf{t_1} \wedge \bigwedge_{i=1}^{m-1} (\forall x_1 \cdots \forall x_i (t_i \longrightarrow (\exists x_{i+1})\mathbf{t_{i+1}}))$$

ψ has the property that any model of ψ contains a substructure isomorphic to $\mathbf{A_0}$.

Assume $\mathbf{B}$ is a finite model of ψ. By Theorem 1, $\mathbf{B}$ is isomorphic to some substructure of $\mathbf{A}$. By including $a_1, \ldots, a_m$ in the isomorphism at the first m steps of construction, we can find $\mathbf{A'}$, which is a substructure of $\mathbf{A}$ that contains $\mathbf{A_0}$. Moreover, this construction ensures that $\mathbf{B} \cong \mathbf{A'}$. We can extend $\mathbf{A'}$ to $\mathbf{A'^*}$ by adjoining to $\mathbf{A}$ the interpretations of $\mathbf{S}$-predicates $\mathbf{S}^*$, restricted to the universe of $\mathbf{A'}$. Because substructures preserve universal statements, we can conclude that

$$\mathbf{A'^*} \models (\forall y_1 \cdots \forall y_n)\theta(x_1, \ldots, x_m, y_1, \ldots, y_n, \mathbf{R}, \mathbf{S})$$

Then, by existential generalization of the first-order variables $x_1, \ldots, x_m$ and the second-order variables in $\mathbf{S}$, $\mathbf{A'^*} \models \Phi$. Then, since $\mathbf{S}$ is no longer free, $\mathbf{A'} \models \Phi$. Finally, because $\mathbf{A'} \cong \mathbf{B}$, $\mathbf{B} \models \Phi$.

We have shown that any finite model of ψ is also a model of Θ, where ψ is a finite conjunction of extension axioms. By Theorem 2, $\mu(\psi) = 1$. Hence $\mu(\Theta) = 1$. $\square$

5 Decidability of the Ackermann Class

We will now present a proof of the decidability of the Ackermann class by constructing a set $\mathbf{P}$ of 1-types that obeys certain properties in regards to their extensions to n-types. For the purpose of clarity, this proof will concern the restricted Ackermann class ($\forall\exists\cdots\exists$) rather than the full Ackermann class ($\exists\cdots\exists\forall\exists\cdots\exists$). Afterwards, I will sketch how the argument can be extended to the full class. Let $\Phi = \forall x_1 \exists x_2 \ldots \exists x_n \varphi$ be a restricted Ackermann formula, with φ quantifier-free.

Theorem 7 (Ackermann Decidability)**.** *Φ is satisfiable if and only if there exists a set $\mathbf{P}$ of 1-types with the following property (♣). For every $\mathbf{t} \in \mathbf{P}$, there is an n-type $\mathbf{t}'$ such that:*

1. *$\mathbf{t}'$ extends $\mathbf{t}$,*

2. *$\mathbf{t}'$ makes φ true. That is, $\mathbf{t}'$ truth-functionally implies φ, and*

3. $[\mathbf{t}'|x_i] \in \mathbf{P} \quad (1 \leq i \leq n)$.

The existence of such a set $\mathbf{P}$ is a finitely verifiable condition, so the claim implies decidability.

Proof. ($\Longrightarrow$) If Φ is satisfiable, then let $\mathbf{M}$ be a model that witnesses that satisfiability. Let $\mathbf{P}$ be the set of 1-types realized in $\mathbf{M}$. $\mathbf{P}$ has property ($\clubsuit$); if $\mathbf{t} \in \mathbf{P}$, let a_1 be such that $\mathbf{tp}_{\mathbf{M}}(a_1) = \mathbf{t}$. Then, since $\mathbf{M}$ is a model for Φ, there are $a_2, \ldots, a_n \in \mathbf{M}$ such that $\mathbf{M} \models \varphi(a_1, \ldots, a_n)$ is true. So, let $\mathbf{t}' = \mathbf{tp}_{\mathbf{M}}(a_1, \ldots, a_n)$. $\mathbf{t}'$ satisfies conditions **1–3** of property ($\clubsuit$).

($\Longleftarrow$) Let $\mathbf{P}$ be a set of 1-types that satisfy conditions **1–3** of property ($\clubsuit$). We will build, by induction, a model $\mathbf{M}$ for Φ such that $\mathbf{P}$ is the set of 1-types realized in $\mathbf{M}$.

Initial Step: Put an object a_1 into the universe of $\mathbf{M}$, denoted $U_{\mathbf{M}}$. Pick some $\mathbf{t} \in \mathbf{P}$ and interpret the predicate letters such that $\mathbf{t} = \mathbf{tp}_{\mathbf{M}}(a_1)$.

Inductive Step: Our inductive hypothesis states that in the model $\mathbf{M}$, as constructed up until now, every 1-type that is realized by an element is, in fact, a member of $\mathbf{P}$.

Let $b \in \mathbf{M}$ be the earliest object put into $\mathbf{M}$ so far such that objects that witness $\exists x_2 \cdots \exists x_n \varphi(x_1/b)$ do not exist in $\mathbf{M}$. Earliest here means placed in the universe at the earliest step in the inductive process. Let $\mathbf{t} = \mathbf{tp}_{\mathbf{M}}(b)$. We are guaranteed to have $\mathbf{t} \in \mathbf{P}$ by our inductive hypothesis. Now, let $\mathbf{t}'$ be the n-type guaranteed by property ($\clubsuit$) of $\mathbf{P}$. Letting $c_2, \ldots, c_n \in \mathbf{M}$ be new objects, extend the interpretation of the predicate letters so that $\mathbf{t} = \mathbf{tp}_{\mathbf{M}}(b, c_2, \ldots, c_n)$. We are guaranteed that this makes φ true by condition 2 of property ($\clubsuit$) on $\mathbf{t}'$, so the c_i's are witnesses to $\exists x_2 \cdots \exists x_n \varphi[x_1/b]$. Moreover, the inductive hypothesis remains true, since the new elements realize 1-types in $\mathbf{P}$, by condition 3 of property ($\clubsuit$).

By continuing along with this process through all the integers, we will create a model for Φ. $\square$

The Problem with Equality: There is a possible problem with our inductive construction if we allow for our types to have formulas including equality. That is, when we are adding elements $c_2, \ldots, c_n$ to the universe, we might have an n-type that has the sentence '$x_i = x_j$', with $i \neq j$, as one of its elements. In this case, we would not want to add in distinct c_i, c_j as we do in our construction, since that would cause our structure to model Φ no longer. A similar problem occurs if the type "says" '$b = c_i$'.

This problem could be solved with additional conditions on our process, but fortunately, there is an easier way. Dreben and Goldfarb ([6], p. 207) proved that for every ψ, an Ackermann first-order sentence, there exists a ψ' Ackermann first-order sentence such that

1. The matrix of ψ' implies $x_i \neq x_j$ for $i \neq j$
2. ψ and ψ' are equivalent over universes of size greater than n.

Since our model will certainly have a universe larger than n, we can ignore the possible problem with equality by assuming we are treating the equivalent ψ' for any ψ that has some statement of equality. A similar fix works for the full Ackermann class.

Note on the Full Ackermann Class: Let

$$\Phi = \exists z_1 \cdots \exists z_m \forall x_1 \exists x_2 \cdots \exists x_n \varphi$$

with φ quantifier free. We can add constants $\alpha_1, \ldots, \alpha_m$ to the language and then instantiate $z_1 \ldots z_m$ by $\alpha_1, \ldots, \alpha_m$. We consider

$$\Phi' = \forall x_1 \exists x_2 \ldots \exists x_n \varphi(z_1/\alpha_1, \ldots, z_m/\alpha_m)$$

which is a restricted Ackermann sentence. As before, whenever $\mathbf{M}$ is a model for Φ', the set of 1-types realized in $\mathbf{M}$ (now in the language expanded by the constants) is a suitable $\mathbf{P}$. We can then go forward with the above proof, with some added conditions on $\mathbf{P}$. We sketch these conditions: if we take a subset of any $\mathbf{t} \in \mathbf{P}$ that consists entirely of formulas that contain only the constants—and not the variable x_1—we always get the same set $\mathbf{s}$. Moreover, for each i (with $1 \leq i \leq m$), there is a $\mathbf{t} \in \mathbf{P}$ such that replacing x_1 in every formula in $\mathbf{t}$ with α_i yields just that set $\mathbf{s}$.

Given this suitable $\mathbf{P}$, we construct a model for Φ' using the above inductive process, but with an alteration at the initial step. First, we introduce objects $a_1, \ldots, a_m \in \mathbf{M}$ to be the values of the constants $\alpha_1, \ldots, \alpha_m$ and interpret the predicates in a way such that $\mathbf{M} \models \mathbf{s}$. Then, from the additional conditions we have placed on $\mathbf{P}$, it follows that $\mathbf{tp}_{\mathbf{M}}(a_i) \in \mathbf{P}$ for $1 \leq i \leq m$. We can then continue with our previous proof of decidability.

6 Finite Controllability of the Ackermann Class

We will now prove that for every first-order Ackermann sentence with a model, there is a finite model. As before, we will treat only the restricted Ackermann class, for the purpose of expository clarity.

Theorem 8 (Ackermann Finite Controllability)**.** *For an Ackermann sentence $\Phi = \forall x_1 \exists x_2 \cdots \exists x_n \varphi(x_1, \ldots, x_n)$, if there is a model for Φ, then there is a finite model $\mathbb{M}$ such that $\mathbb{M} \models \Phi$.*

Proof. Let $\mathbf{B}$ be a model for Φ. Since there is a model for Φ, by Theorem 7, there is a set $\mathbf{P}$ that has property ($\clubsuit$). Now we define a mapping from $\mathbf{P}$ to the set of n-types guaranteed to exist by property ($\clubsuit$).

Definition 11. *For $\mathbf{t} \in \mathbf{P}$, let $\eta(\mathbf{t}) = \mathbf{t}'$, where $\mathbf{t}'$ is the n-type that witnesses property* ($\clubsuit$)*.*

Now, let $\mathbf{Q}$ be the range of η, and set

$$\mathbf{I} = \{0, 1, 2\} \times \mathbf{Q} \times \{2, \ldots, n\}.$$

Now, we will define a structure $\mathbf{M}$ with its universe equal to $\mathbf{I}$ by way of two rules. Elements of the structure $\mathbf{M}$ will be ordered triples, $a = \langle \delta, \mathbf{s}, i \rangle$.

Rule 1: For all $a \in \mathbf{M}$, set $\mathbf{tp}_{\mathbf{M}}(a) = [\pi_2(a)|x_{\pi_3(a)}]$.

Note that it follows from Rule 1 and Clause 3 of condition (♣) that $\mathbf{tp}_{\mathbf{M}}(a) \in \mathbf{P}$.

Rule 2: For any $b, c_2, \ldots, c_n$ with the following three properties:

(I) $\pi_1(c_2) \equiv \ldots \equiv \pi_1(c_n) \equiv \pi_1(b) + 1 \pmod 3$,

(II) $\pi_3(c_i) = i$, for $2 \leq i \leq n$, and

(III) There is some some $\mathbf{s} \in \mathbf{Q}$ such that

 (i) $\pi_2(c_2) = \ldots = \pi_2(c_n) = \mathbf{s}$

 (ii) $[\mathbf{s}|x_1] = \mathbf{tp}_{\mathbf{M}}(b)$,

then,
set $\mathbf{tp}_{\mathbf{M}}(b, c_2, \ldots, c_n) = \mathbf{s}$.

Insofaras the interpretations of predicate letters are not assigned by Rule 2, let them be arbitrary, but such that they do not conflict with the interpretations set by Rule 2.

It must be shown that Rules 1 and 2 engender no ambiguity in the assignment of types. Assuming this has been demonstrated, we will have constructed, by way of Rules 1 and 2, a *finite* model $\mathbb{M}$ with universe $\mathbf{I}$ that makes φ true. Let $b \in \mathbf{I}$. There are elements of the model $c_2 = \langle j, \eta(\mathbf{t}), 2 \rangle, \ldots, c_n = \langle j, \eta(\mathbf{t}), n \rangle$, where $j \equiv \pi_1(b) + 1$

(mod 3) and $\mathbf{t} = \mathbf{tp_M}(b)$, such that $\mathbf{tp_M}(b, c_2, \ldots, c_n) = \eta(\mathbf{t})$. Since $\eta(\mathbf{t})$ implies φ, $\mathbf{M} \models \varphi(b, c_2, \ldots, c_n)$. □

We must now prove that Rules 1 and 2 are not ambiguous. For some atomic formula $\mathcal{R} \in \mathbf{R}$, the relational vocabulary, we might have assigned two different truth values in two different n-types to the same $d, e \in \mathbf{M}$, such that $\mathcal{R}de$ is true in one type, but false in another. We will prove that this cannot be the case. For simplicity, we will concern ourselves with only dyadic predicates; the proof for n-ary predicates proceeds along the same lines, but complicated bookkeeping of indices obscures the proof. First, we know that $d \neq e$, for if $d = e$, $\mathcal{R}de$ would be set by Rule 1, which is clearly unambiguous. We have two possible cases of ambiguity to address:

Case 1: $\pi_1(d) = \pi_1(e)$

For this assignment to be ambiguous, it would have to be the case that $\{d, e\} \subset \{c_2, \ldots, c_n\}$ and $\{d, e\} \subset \{c'_2, \ldots, c'_n\}$. Then, because they fall under the aegis of Rule 2,

$$\pi_2(d) = \pi_2(e)$$

$$\pi_3(d) \neq \pi_3(e)$$

Because their π_3 values are distinct, we know that d must appear in ith position in both sets, while e must appear in the jth position. That is, $d = c_i = c'_i$ where $i = \pi_3(d)$, and $e = c_j = c'_j$ where $j = \pi_3(e)$. Then, the truth value of $\mathcal{R}de$ is given to be true or false depending on whether $\pi_2(d)$ contains $\mathcal{R}x_ix_j$ or $\neg\mathcal{R}x_ix_j$, respectively, and there is no ambiguity.

Case 2:

$$\pi_1(d) \neq \pi_1(e)$$

Such a situation would mean either $\pi_1(e) = \pi_1(d) + 1$ (mod 3) or $\pi_1(d) = \pi_1(e) + 1$

(mod 3). Without loss of generality, we can assume the former, which means that $d = b = b'$ and $e = c_i = c_i'$, where $i = \pi_3(e)$. By the same argument as in Case 1, the truth value for $\mathcal{R}de$ is given by whether $\pi_2(e)$ contains $\mathcal{R}x_1x_i$ or $\neg\mathcal{R}x_1x_i$, and there is no ambiguity.

7 0-1 Law for the Σ_1^1 Ackermann Class

Before approaching the details, we will give a survey of our proof of the 0-1 law. We have already shown that the negative direction of a Transfer Theorem holds for all Σ_1^1 formulas by Corollary 3. That is, if we let Φ be as follows,

$$\Phi = \exists \mathbf{S} \forall x_1 \exists x_2 \cdots \exists x_n \varphi(x_1, x_2, \ldots, x_n, \mathbf{R}, \mathbf{S}),$$

we know that if $\mathbf{A} \models (\neg\Phi)$, then $\mu(\neg\Phi) = 0$. To prove a 0-1 law holds for the Σ_1^1 Ackermann class, it suffices to prove that if $\mathbf{A} \models \Phi$, then $\mu(\Phi) = 1$. Note that Φ has predicates both in $\mathbf{R}$, which are free and not quantified over, and in $\mathbf{S}$, which are quantified over. This was the case for the Bernays-Schönfinkel class as well. However, because the earlier proof did not involve our differentiating between the two vocabularies, we elided the differences from our discussion. We will unavoidably be involved with these differences in our current proof, so we will have to refine our notions of type and its related concepts.

We define an $\mathbf{R}$-structure as a structure that has interpretations of only the $\mathbf{R}$-predicates. In contrast, an $(\mathbf{R}, \mathbf{S})$-structure will be a structure with interpretations of both $\mathbf{R}$ and $\mathbf{S}$ predicates. Because the $\mathbf{S}$-predicates are quantified over, we are able to define an interpretation of them to suit our needs. We will do this via two rules, similar to those used in the finite controllability proof for the Ackermann Class. This deliberate method of constructing the $(\mathbf{R}, \mathbf{S})$-model of Φ will avoid the probabilistic methods needed in the Kolaitis and Vardi proof [18]. However, due to our use of both $\mathbf{R}$-types and $(\mathbf{R},\mathbf{S})$-types, we must extend our notation of types to avoid confusion. An $\mathbf{R}$-n-type is an n-type for the vocabulary $\mathbf{R}$, whereas an $(\mathbf{R},\mathbf{S})$-n-type is an n-type for the vocabulary $(\mathbf{R},\mathbf{S})$ (cf. Definition 6).

Definition 12. *If* $\mathbf{M}$ *is an* $\mathbf{R}$*-structure or an* $(\mathbf{R},\mathbf{S})$*-structure, then* $\mathbf{tp}^{\mathbf{R}}_{\mathbf{M}}(a_1,\ldots,a_n)$ *is the unique* $\mathbf{R}$*-*n*-type realized by* $\langle a_1,\ldots,a_n\rangle$ *in* $\mathbf{M}$*. If* $\mathbf{M}$ *is an* $(\mathbf{R},\mathbf{S})$*-structure,* $\mathbf{tp}^{(\mathbf{R},\mathbf{S})}_{\mathbf{M}}(a_1,\ldots,a_n)$ *is the unique* $(\mathbf{R},\mathbf{S})$*-*n*-type realized by* $\langle a_1,\ldots,a_n\rangle$ *in* $\mathbf{M}$*.*

Definition 13 (Reduct)**.** *We say an* $\mathbf{R}$*-type* $\mathbf{t}^{\mathbf{R}}$ *is the* ***reduct*** *of an* $\mathbf{R},\mathbf{S}$*-type* $\mathbf{t}^{(\mathbf{R},\mathbf{S})}$*, if* $\mathbf{t}^{\mathbf{R}} \subseteq \mathbf{t}^{(\mathbf{R},\mathbf{S})}$*,* $\mathbf{t}^{\mathbf{R}}$ *has no* $\mathbf{S}$*-predicates, and for every* $\mathbf{R}$*-formula* f *in* $\mathbf{t}^{(\mathbf{R},\mathbf{S})}$*,* f *is also in* $\mathbf{t}^{\mathbf{R}}$*.*

$[\mathbf{t}|\,\mathbf{R}]$ denotes a reduct of $\mathbf{t}$ to $\mathbf{R}$. Essentially, the reduct is obtained by taking an $(\mathbf{R},\mathbf{S})$-type and throwing out any formulae with an $\mathbf{S}$-predicate.

Returning to the Transfer Theorem, the direction that we must prove is:

$$\text{If } \mathbf{A} \models \Phi, \text{ then } \mu(\Phi) = 1$$

Assume that $\mathbf{A} \models \Phi$. We will define a condition $(\bigstar)$, and show that $(\bigstar)$ is almost surely true for C, the class of all finite $\mathbf{R}$-models. That is, $\mu(\bigstar) = 1$. Given a finite $\mathbf{R}$-structure $\mathbf{M}$ with condition $(\bigstar)$ such that $\mathbf{M} \models \Phi$, we will show that it is possible to extend $\mathbf{M}$ to an $(\mathbf{R},\mathbf{S})$-structure $\mathbf{M}^*$ by interpreting the $\mathbf{S}$-predicates in such a way that $\mathbf{M}^* \models \forall x_1 \exists x_2 \cdots \exists x_n \varphi(x_1,\ldots,x_n,\mathbf{R},\mathbf{S})$. Thus, $\mu(\Phi) = 1$.

This proof will be for the restricted Ackermann sentence that has no existential quantifiers before the universal quantifier. Later, we will illustrate how the proof can be adapted to the unrestricted class.

Claim (Transfer Theorem for Σ^1_1 Ackermann Class). *If* $\mathbf{A} \models \Phi$*, then* $\mu(\Phi) = 1$.

Proof. We want to define a way of interpreting $\mathbf{S}$-predicates for the Ackermann sentence that makes φ true for both the $\mathbf{R}$- and the $\mathbf{S}$-predicates. So, since $\mathbf{A} \models \Phi$, let $\mathbf{A}^*$ be the extension of $\mathbf{A}$ such that

$$\mathbf{A}^* \models \forall x_1 \exists x_2 \cdots \exists x_n \varphi(x_1, x_2, \ldots, x_n, \mathbf{R}, \mathbf{S}).$$

We can define $\mathbf{P}$ as the set of 1-types realized in $\mathbf{A}^*$. $\mathbf{P}$ then is a set of "proper" 1-types in the vocabulary $(\mathbf{R}, \mathbf{S})$. They are proper in that they are the 1-types that do not contradict the formula $\exists x_2 \cdots \exists x_n \varphi(x_1, x_2, \ldots, x_n, \mathbf{R}, \mathbf{S})$. Formally, $\mathbf{P} = \{\mathbf{tp}^{(\mathbf{R},\mathbf{S})}_{\mathbf{A}^*}(\mathbf{a}) | \, \mathbf{a} \in \mathbf{A}\}$.

Since $\mathbf{A}^* \models \forall x_1 \exists x_2 \cdots \exists x_n \varphi(x_1, x_2, \ldots, x_n, \mathbf{R}, \mathbf{S})$, a first-order Ackermann sentence, the set $\mathbf{P}$ will have property ($\clubsuit$). Furthermore, since $\mathbf{A}$ is the countable random $\mathbf{R}$-structure, for every $\mathbf{R}$-1-type $\mathbf{t_0} \in \mathbf{A}$, there will be an $(\mathbf{R}, \mathbf{S})$-1-type $\mathbf{t} \in \mathbf{P}$ such that $[\mathbf{t}|\mathbf{R}] = \mathbf{t_0}$. We are guaranteed such a $\mathbf{t}$ because there is some $a \in \mathbf{A}$ such that $\mathbf{t_0} = \mathbf{tp}^{\mathbf{R}}_{\mathbf{M}}(a)$, and we can take $\mathbf{t}$ to be the $(\mathbf{R}, \mathbf{S})$-n-type such that $\mathbf{t} = \mathbf{tp}^{(\mathbf{R},\mathbf{S})}_{\mathbf{A}^*}(a)$. Throughout the proof, $\mathbf{M}$ will be some $\mathbf{R}$-structure with universe $\{1, \ldots, m\}$ that has property ($\bigstar$), and $\mathbf{M}^*$ will be an $(\mathbf{R}, \mathbf{S})$-structures we will construct by interpreting $\mathbf{S}$-predicates.

We now define $\eta(\mathbf{t})$, the function that maps a 1-type, $\mathbf{t} \in \mathbf{P}$, to an n-type that makes the matrix true, with the restriction to the first term equal to $\mathbf{t}$ and the restriction to the ith term equal to some other element of $\mathbf{P}$. This definition is the same as definition 11, but defined over $(\mathbf{R}, \mathbf{S})$-predicates instead of merely $\mathbf{R}$-predicates. Now we will give the formal definition.

Definition 14. *Let* $\mathbf{t} \in \mathbf{P}$*. Since* $\mathbf{P}$ *has property* ($\clubsuit$)*, let* $\mathbf{t}'$ *be the* $(\mathbf{R}, \mathbf{S})$*-*n*-type given* *by condition ($\clubsuit$), and define $\eta(\mathbf{t}) = \mathbf{t}'$, with the following properties:*

(i) $\eta(\mathbf{t}) \models \varphi$

(ii) $[\eta(\mathbf{t})|x_1] = \mathbf{t}$

(iii) $[\eta(\mathbf{t})|x_i] \in \mathbf{P}$*, for* $i \in \{2, \ldots, n\}$

Now define $\mathbf{Q} = \{\eta(\mathbf{t}) | \mathbf{t} \in \mathbf{P}\}$ and the set

$$\mathbf{I} = \{0, 1, 2\} \times \mathbf{Q} \times \{2, \ldots, n\}$$

Since $\mathbf{Q}$ is the set of $(\mathbf{R}, \mathbf{S})$-n-type extensions, the cardinality of $\mathbf{Q}$, q, is finite and depends only on Φ. Let $l = 3q(n-1)$. Thus, $|\mathbf{I}| = l$. Given $\mathbf{M}$ with universe equal to $\{1, \ldots, m\}$, let $h : \{0, \ldots, l-1\} \longrightarrow \mathbf{I}$ be a one-to-one and onto function. Then, for each $j \in \{1, \ldots, m\}$, let $[j]$ be the least non-negative remainder of j mod l.

We now define the following indices:

$$index_1(j) = \pi_1\left(h([j])\right)$$
$$index_2(j) = \pi_2\left(h([j])\right)$$
$$index_3(j) = \pi_3\left(h([j])\right)$$

Thus, for any triple $\langle \delta, \mathbf{s}, i \rangle \in \mathbf{I}$, there will be roughly $\frac{m}{l}$ elements $a \in \mathbf{M}$ with $index_1(a) = \delta$, $index_2(a) = \mathbf{s}$, and $index_3(a) = i$. We can now give condition ($\bigstar$) and prove that it holds in almost all finite $\mathbf{R}$-models.

Condition ($\bigstar$): For any $b \in \mathbf{M}$, for any $(\mathbf{R}, \mathbf{S})$-n-type $\mathbf{s} \in \mathbf{Q}$, we can find $c_2, \ldots, c_n$, elements of $\mathbf{M}$, with the following properties, for $2 \leq i \leq n$:

(i) $index_1(c_2) \equiv \ldots \equiv index_1(c_n) \equiv index_1(b) + 1 \pmod 3$

(ii) $index_2(c_2) = \ldots = index_2(c_n) = \mathbf{s}$

(iii) $index_3(c_i) = i$

(iv) $[\mathbf{s}|\mathbf{R}] = \mathbf{tp}^{\mathbf{R}}_{\mathbf{M}}(b, c_2, \ldots, c_n)$

Lemma 1. *Condition (★) holds for almost every finite structure. That is, for finite structures of size m, $\mu\big((\bigstar)\big) = 1$.*

Proof. For any $b \in \mathbf{M}$ and for any $\mathbf{s}$, we need to find the probability that condition (★) holds and prove that this quantity tends toward 1 as the size of $\mathbf{M}$ increases. For a given $\langle b, c_2, \ldots, c_n\rangle$, the probability that the n-tuple satisfies requirement (iv) of condition (★) is greater than or equal to $\frac{1}{q}$, where $q = |\mathbf{Q}|$.

This means that the probability that a given $\langle b, c_2, \ldots, c_n\rangle$ fails is $\left(1 - \frac{1}{q}\right)$. For a given $b \in \mathbf{M}$ and $\mathbf{s} \in \mathbf{Q}$, we want to find the number of disjoint possibilities for sets of $c_2, \ldots, c_n$ that satisfy conditions (i)-(iii). This way, the set fails to satisfy condition (★) only if it fails to satisfy condition (iv). Given two disjoint sets, their respective failures of condition (★) by way of failing condition (iv) are independent events. Multiplying their probabilities together will give us an upper bound on the probability that all such $c_2, \ldots, c_n$ fail to satisfy condition (★). Thus, we want a lower bound on the number of possible choices for disjoint $c_2, \ldots, c_n$.

Claim. *For a given $b \in \mathbf{M}$ and $\mathbf{s} \in \mathbf{Q}$, the number of possible choices for disjoint $c_2, \ldots, c_n$ is at least $\frac{m}{l}$, with $m = |\mathbf{M}|$ and $l = |\mathbf{I}|$.*

Proof. Let $j \leq \lfloor \frac{m}{l} \rfloor$, that is the integer part of $\frac{m}{l}$. We will show that there is at least one suitable $(n-1)$-tuple $\langle c_2, \ldots, c_n\rangle$ in $\{j(l+1), \ldots, j(l)\}$.

Let $\delta \equiv index_1(b) + 1 \pmod 3$, and for $2 \leq i \leq n$, let $p_i = h^{-1}(\langle \delta, \mathbf{s}, i\rangle)$. Now we define $c_i = jl + p_i$. These $c_2, \ldots, c_n$ are suitable potential witnesses.

□

Given this lower bound on the number of choices, we have the following upper bound, for a given $b \in \mathbf{M}$ and $\mathbf{s} \in \mathbf{Q}$:

$$\text{Prob[all } c_2, \ldots, c_n \text{ fail to satisfy condtion } (\bigstar)] \leq \left(1 - \frac{1}{q}\right)^{\frac{m}{l}}$$

Since the number of possible choices for b is $m = |\mathbf{M}|$, and the number of possible choices for $\mathbf{s}$ is $q = |\mathbf{Q}|$,

$$\text{Prob[condition } (\bigstar) \text{ fails for the model } \mathbf{M}] \leq m \left(1 - \frac{1}{q}\right)^{\frac{m}{l}}$$

Since q and l are fixed based on the formula, which is independent of m, the quantity $m\left(1 - \frac{1}{q}\right)^{\frac{m}{l}}$ will tend to zero as m tends to infinity. Therefore, condition $(\bigstar)$ is almost always true on finite models.

$\square$

Given an $\mathbf{R}$-structure that has condition $(\bigstar)$, $\mathbf{M} \in C$, we will show how to extend $\mathbf{M}$ to an $(\mathbf{R}, \mathbf{S})$-structure $\mathbf{M}^*$ that is a model for $\forall x_1 \exists x_2 \cdots \exists x_n \varphi(x_1, x_2, \ldots, x_n, \mathbf{R}, \mathbf{S})$. Such an extension will prove that $\mathbf{M} \models \Phi$. Then, since condition $(\bigstar)$ is almost surely true for finite models, $\mu(\Phi) = 1$. We will extend $\mathbf{M}$ to $\mathbf{M}^*$ by way of two rules, similar to the rules we established for the finite controllability proof of the Ackermann class, which will give interpretations $\mathbf{S}$ through the assignment of types. For some element $c \in \mathbf{M}$, we first define a rule for extending the $\mathbf{R}$-1-types to $(\mathbf{R}, \mathbf{S})$-1-types such that the extensions are in $\mathbf{P}$.

Rule 1: Let $c \in \mathbf{M}$ and let $\mathbf{t_0} = [index_2(c)|x_{index_3(c)}]$. If $[\mathbf{t_0}|\mathbf{R}] = \mathbf{tp}^{\mathbf{R}}_{\mathbf{M}}(c)$, then let $\mathbf{tp}^{(\mathbf{R},\mathbf{S})}_{\mathbf{M}^*}(c) = \mathbf{t_0}$.

If $[\mathbf{t_0}|\mathbf{R}] \neq \mathbf{tp}^{\mathbf{R}}_{\mathbf{M}}(c)$, then pick an arbitrary $\mathbf{t} \in \mathbf{P}$ such that $[\mathbf{t}|\mathbf{R}] = \mathbf{tp}^{\mathbf{R}}_{\mathbf{M}}(c)$. Then set $\mathbf{tp}^{(\mathbf{R},\mathbf{S})}_{\mathbf{M}^*}(c) = \mathbf{t}$.

In establishing this rule, we are concerned only with the c's such that $[\mathbf{t_0}(c)|\,\mathbf{R}] = \mathbf{tp}^{\mathbf{R}}_{\mathbf{M}}(a)$. In defining the $(\mathbf{R},\mathbf{S})$-types for other c's randomly, we are merely making sure that nothing goes "wrong" with their extensions to the $(\mathbf{R},\mathbf{S})$-structure; that is, we ensure that the types are always in $\mathbf{P}$.

Now, we establish a second rule for setting the n-types in $\mathbf{M}^*$, given n elements of $\mathbf{M}$ that stand in certain relations to each other.

Rule 2: For any b, $c_2, \ldots, c_n \in \mathbf{M}$ with the following properties:

(I) $index_1(c_2) \equiv \ldots \equiv index_1(c_n) \equiv index_1(b) + 1 \pmod 3$

(II) there exists a $\mathbf{t} \in \mathbf{P}$ such that

- (a) $\mathbf{tp}^{(\mathbf{R},\mathbf{S})}_{\mathbf{M}^*}(b) = [\eta(\mathbf{t})|x_1]$
- (b) $\mathbf{tp}^{(\mathbf{R},\mathbf{S})}_{\mathbf{M}^*}(c_i) = [\eta(\mathbf{t})|x_i]$
- (c) $index_2(c_i) = \eta(\mathbf{t})$

(III) $index_3(c_i) = i$

(IV) $\mathbf{tp}^{\mathbf{R}}_{\mathbf{M}^*}(b, c_2, \ldots, c_n) = [\eta(\mathbf{t})|\mathbf{R}]$,

Set $\mathbf{tp}^{\mathbf{R},\mathbf{S}}_{\mathbf{M}^*}(b, c_2, \ldots, c_n) = \eta(\mathbf{t})$. Let any other extension be arbitrary.

The spectre of ambiguous assignment might again worry us, but the same argument we made earlier in the finite controllability proof is applicable to the current case. We have constructed $\mathbf{M}^*$ as an extension of $\mathbf{M}$ by Rules 1 and 2.

Claim 1. *For an extension $\mathbf{M}^*$ of $\mathbf{M}$ by way of Rules 1 and 2,*

$$\mathbf{M}^* \models \forall x_1 \exists x_2 \cdots \exists x_n \varphi(x_1, x_2, \ldots, x_n, \mathbf{R}, \mathbf{S})$$

Proof. Let $b \in \mathbf{M}^*$ and $\mathbf{t_0} = \mathbf{tp}^{(\mathbf{R},\mathbf{S})}_{\mathbf{M}^*}(b)$. Note that $\mathbf{t_0} \in \mathbf{P}$. Now instantiate in such a way as to apply condition ($\bigstar$) according to the following method:

Let $i \equiv index_1(b) + 1 \pmod 3$

Let $\mathbf{t} = [\eta(\mathbf{t_0})|\mathbf{R}]$

Let $\mathbf{s} = \eta(\mathbf{t_0})$

By condition ($\bigstar$), we have $c_2, \ldots, c_n \in \mathbf{M}$ with the following properties:

(i) $index_1(c_1) \equiv \ldots \equiv index_1(c_n) \equiv 1 + index_1(b) \pmod 3$

(ii) $index_2(c_2) = \ldots = index_2(c_n) = \mathbf{s} = \eta(\mathbf{t_0})$

(iii) $index_3(c_i) = i$

(iv) $\mathbf{t} = [\eta(\mathbf{t_0})|\ \mathbf{R}] = \mathbf{tp}^{\mathbf{R}}_{\mathbf{M}}(b, c_2, \ldots, c_n)$.

Notice that $\mathbf{tp}^{\mathbf{R}}_{\mathbf{M}}(b, c_2, \ldots, c_n) = [\eta(\mathbf{t_0})|\mathbf{R}]$. This means that

$$\begin{aligned}
\mathbf{tp}^{\mathbf{R}}_{\mathbf{M}}(c_{index_3(c_i)}) &= \mathbf{tp}^{\mathbf{R}}_{\mathbf{M}}(c_i) \\
&= [\eta(\mathbf{t_0})|\mathbf{R}|x_{index_3(c_i)}] \\
&= [\eta(\mathbf{t_0})|x_{index_3(c_i)}|\mathbf{R}] \\
&= [index_2(c_i)|x_{index_3(c_i)}|\mathbf{R}]
\end{aligned}$$

So, since $\mathbf{tp}^{\mathbf{R}}_{\mathbf{M}}(c_{index_3(c_i)}) = [index_2(c_{index_3(c_i)})|\mathbf{R}]$, Rule 1 is invoked, and $\mathbf{tp}^{(\mathbf{R},\mathbf{S})}_{\mathbf{M}^*}(c_i) = [index_2(c_i)|x_{index_3(c_i)}]$. That means that the conditions of Rule 2 hold, with $\mathbf{t_0}$ as the witness. That is,

(I) $index_1(c_i) \equiv index_1(b) + 1 \pmod 3$

(II) $\mathbf{t_0} \in \mathbf{P}$ such that

  (a) $\mathbf{tp}^{\mathbf{R},\mathbf{S}}_{\mathbf{M}^*}(b) = [\eta(\mathbf{t_0})|\ x_1]$

(b) $\mathbf{tp}^{\mathbf{R},\mathbf{S}}_{\mathbf{M}^*}(c_i) = [\eta(\mathbf{t_0}) |\, x_i]$

(c) $\pi_2(c_i) = \eta(\mathbf{t_0})$

(III) $index_3(c_i) = i$

(IV) $\mathbf{tp}^{\mathbf{R}}_{\mathbf{M}^*}(b, c_2, \ldots, c_n) = [\eta(\mathbf{t_0}) |\, \mathbf{R}]$,

So,

$$\begin{aligned}\mathbf{tp}^{(\mathbf{R},\mathbf{S})}_{\mathbf{M}^*}(b, c_2, \ldots, c_n) &= \eta(\mathbf{t_0}) \\ &= \eta\left(\mathbf{tp}^{(\mathbf{R},\mathbf{S})}_{\mathbf{M}^*}(b)\right)\end{aligned}$$

Thus, for any $b \in \mathbf{M}$, there are $c_2, \ldots, c_n$ such that $\mathbf{M}^* \models \varphi(b, c_2, \ldots, c_n, \mathbf{R}, \mathbf{S})$. Therefore,

$$\mathbf{M}^* \models \forall x_1 \exists x_2 \cdots \exists x_n \varphi(x_1, x_2, \ldots, x_n, \mathbf{R}, \mathbf{S})$$

□

Given $\mathbf{R}$-predicates and the fact that $\mathbf{A} \models \exists \mathbf{S}\, \forall x_1 \exists x_2 \cdots \exists x_n \varphi(x_1, x_2, \ldots, x_n, \mathbf{R}, \mathbf{S})$, we have formulated an interpretation of $\mathbf{S}$ such that $\mathbf{M}^*$, a finite model where condition (★) holds, satisfies Φ. However, condition (★) holds for nearly all finite models, so $\mu(\Phi) = 1$.

□

Note on the Full Ackermann Class: We have proven the 0-1 law holds for the restricted class, without any initial first-order existential quantifiers. To prove the law for the full class, we let $\Phi = \exists \mathbf{S} \exists z_1 \cdots \exists z_k \forall x_1 \exists x_2 \cdots \exists x_n \varphi$ and assume

$$\mathbf{A} \models \exists \mathbf{S} \exists z_1 \cdots \exists z_k \forall x_1 \exists x_2 \cdots \exists x_n \varphi(z_1, \ldots, z_k, x_1, \ldots, x_n, \mathbf{R}, \mathbf{S})$$

We have to consider 1-types and n-types in the expanded vocabulary that now contains constants, call them $(\alpha_1, \ldots, \alpha_k)$. Our first step, however, remains the same, in that we define an $\mathbf{A}^*$, an extension of $\mathbf{A}$, such that

$$\mathbf{A}^* \models \exists z_1 \cdots \exists z_k \forall x_1 \exists x_2 \cdots \exists x_n \varphi(z_1, \ldots, z_k, x_1, \ldots, x_n, \mathbf{R}, \mathbf{S}).$$

We now fix a $d_1, \ldots, d_k \in \mathbf{A}^*$ such that

$$\mathbf{A}^* \models \forall x_1 \exists x_2 \cdots \exists x_n \varphi(d_1, \ldots, d_k, x_1, \ldots, x_n, \mathbf{R}, \mathbf{S}).$$

Instead of the $\mathbf{P}$ from the restricted proof, we define a $\mathbf{P}^*$ to be the set of all $(\mathbf{R}, \mathbf{S})$-1-types in this extended language that are realized in $\mathbf{A}^*$ such that $\alpha_1, \ldots, \alpha_k$ are interpretations of $d_1, \ldots, d_k$. Our mapping $\eta(\mathbf{t})$ takes these 1-types to $(\mathbf{R}, \mathbf{S})$-n-types in the extended language. As before, we let $\mathbf{Q}$ be the range of η and $\mathbf{I} = \{0, 1, 2\} \times \mathbf{Q} \times \{2, \ldots, n\}$.

Now we define two conditions, $(\bigstar_1)$ and $(\bigstar_2)$, that will take the place of our $(\bigstar)$ above. They are:

$(\bigstar_1)$ There are $a_1, \ldots, a_k \in \mathbf{M}$ such that $\mathbf{tp}^{\mathbf{R}}_{\mathbf{M}}(a_1, \ldots, a_l) = \mathbf{tp}^{\mathbf{R}}_{\mathbf{A}}(d_2, \ldots, d_k)$.

$(\bigstar_2)$ Let $a_1, \ldots, a_k$ be the earliest elements that fulfill $(\bigstar_1)$. Then, for any $b \in \mathbf{M}$, for any $(\mathbf{R}, \mathbf{S})$-n-type $\mathbf{s} \in \mathbf{Q}$, we can find some elements of the model $c_2, \ldots, c_n$ with the following properties:

(i) $c_i \neq a_j$ for $1 \leq j \leq k$.

(ii) $index_1(c_i) \equiv index_1(b) + 1 \pmod 3$

(iii) $index_2(c_i) = \mathbf{s}$

(iv) $index_3(c_i) = i$

(v) $[\mathbf{s}|\mathbf{R}] = \mathbf{tp}^{\mathbf{R}}_{\mathbf{M}}(b, c_2, \ldots, c_n)$, when $\alpha_1, \ldots, \alpha_k$ are interpreted as $a_1, \ldots, a_k$

where $2 \leq i \leq n$.

We can make a probability argument for ($\bigstar_1$) that shows that the probability is overwhelming for models of large enough size. Then, when ($\bigstar_1$) holds, we set

$$\mathbf{tp}_{\mathbf{M}^*}^{(\mathbf{R},\mathbf{S})}(a_1, \ldots, a_k) = \mathbf{tp}_{\mathbf{A}^*}^{(\mathbf{R},\mathbf{S})}(d_1, \ldots, d_k)$$

We would then make another probability argument, similar to the one for ($\bigstar$), to show that ($\bigstar_2$) holds for almost all finite models, and proceed along a path similar to the proof for the restricted class. One caution should be taken, however, with the analogs to Rules 1 and 2. We must restrict the domain of the new Rule 1 to $c \neq a_1, \ldots, a_k$. Moreover, we must restrict the domain of Rule 2 to $c_2, \ldots, c_n$ disjoint from $a_2, \ldots, a_k$. By so restricting their domains, we prevent them from reassigning the $a_2, \ldots, a_k$.

8 Advantages of Our Proof

In order to illustrate the advantages of our proof of a 0-1 law for the Σ_1^1 Ackermann class over the Kolaitis and Vardi approach, we will briefly give an overview of their proof, without dwelling on unnecessary details. This survey will highlight how our refinement is a simplification.

The Kolaitis and Vardi proof proceeds in much the same direction as ours: it looks to prove a Transfer Theorem for the Σ_1^1 Ackermann Class. So, it assumes $\mathbf{A} \models \Phi$, and shows that $\mu(\Phi) = 1$. To do this, they give three lemmas, finally combining these three smaller results together to show a Transfer Theorem.

- **(Lemma 1)** They define a syntactic condition essentially the same as our condition ($\clubsuit$), and show that it holds for an Ackermann sentence Φ whenever Φ is true on $\mathbf{A}$.

- **(Lemma 2)** They define a "richness" property E_s, for $s \geq 1$ for (some) finite structures over $\mathbf{R}$, then show that $\mu(E_s) = 1$ for any $s \geq 1$.

- **(Lemma 3)** They show that for Φ, a Σ_1^1 Ackermann sentence for which condition ($\clubsuit$) holds, they can pick an $s \in \mathbb{N}$ such that for sufficiently large m, the sentence Φ is true of finite structures of cardinality m over $\mathbf{R}$ that have property E_s.

Since property E_s is almost always true, $\mu(\Phi) = 1$, and the Transfer Theorem holds.

The central difference and advantage of our proof is the relative simplicity of our probability argument compared to those used to push through lemmas 2 and 3. In our proof, for any $b \in \mathbf{M}$, we want to ensure that there are q-many $(n-1)$-tuples $\langle c_2, \ldots, c_n \rangle$ that act as potential witnesses for b. That is, they are elements of the model such that, potentially, $\mathbf{M}^* \models \varphi(b, c_2, \ldots, c_n)$. By ensuring enough *potential* witnesses, we ensure that there will be an $(n-1)$-tuple that is *actually* a set of witnesses for the given b.

We guarantee a sufficiently large space of witnesses by having an n-tuple for every $\mathbf{s} \in \mathbf{Q}$. However, since the size of this potential witness space is fixed independently of the size of $\mathbf{M}$, the probability argument for the existence of such a space—our condition ($\bigstar$)—is quite simple.

In contrast, Kolaitis and Vardi peg their potential witnesses to the size of $\mathbf{M}$. Specifically, in their proof of Lemma 3, in order to show that $\mathbf{M} \models \Phi$, they use a finite random structure argument, due to Gurevich and Shelah [14], to obtain suitable interpretations of the $\mathbf{S}$-predicates. Because of this argument, they require that there be at least $\lceil \sqrt{m}\, \rceil$ potential witnesses for any b. This requirement necessitates that their condition E_s be much more powerful, thereby requiring a far more complicated probability argument due to Chernoff [4] in order to show that E_s is almost surely true for finite models. Because we define the interpretations of the **S**-predicates directly through Rules 1 and 2, we avoid such complications and the need for such powerful probability arguments.

9 Remarks on the Σ_1^1 Gödel Class and 0-1 Laws

We will here include a few concluding remarks concerning the Le Bars [1] proof that a 0-1 law cannot hold for the Σ_1^1 Gödel class (without equality), whereby he disproves the conjecture made by Kolaitis and Vardi [16]. He proves this result by using a notion from graph theory, the kernel, which we now define:

Definition 15 (Kernel). *A set of vertices U of a directed graph H is a* ***kernel*** *if there is no arc (directed edge) inside U and for each vertex that is in H but not in U, there is an arc that goes to U.*

Let $\mathcal{K}$ be the following Σ_1^1 Gödel sentence ($\forall\forall\exists$):

$$\exists U\,((\forall x \forall y((Ux \wedge Uy) \rightarrow Rxy)) \wedge (\forall x \exists y(\neg Ux \rightarrow Uy \wedge Rxy)))$$

Note that $\mathcal{K}$ holds if and only if the structure, when considered as a directed graph, has a kernel. $\mathcal{K}$ is true on the infinite random graph. There is, in fact, an infinite kernel. To see this, we need only remember that a fundamental notion of the infinite random graph is the fact that, in the graph, anything that can happen, does. Suppose for a contradiction, that there is a finite kernel of size k. For any given element not in the kernel, the probability that it has no arc into the kernel is $\left(\frac{1}{2}\right)^k$. The probability that it has such an arc, then, is $1 - \left(\frac{1}{2}\right)^k$. For a structure of size m, the probability that all elements not in the kernel have an arc into the kernel is $\left(1 - \left(\frac{1}{2}\right)^k\right)^{m-k}$. Since k is fixed, as m increases, this quantity will tend toward zero. Thus, with probability

1, we have omitted some element from the kernel that should be included. Thus, the kernel must be infinite.

In finite graphs, however, it is unlikely to have a kernel that is large relative to the size of the graph, since the probability of having k elements without an arc between them is $\left(\frac{1}{4}\right)^{k(k-1)}$. Moreover, it is also improbable to have a kernel that is too small, relative to the size of the graph, since the probability that every element outside of the kernel has an arc into a small set is itself small. It is the case that $\mu(\mathcal{K}) = 1$, but more specifically, there is asymptotic probability of 1 that there exists a kernel of a certain size relative to the size of the graph. That is, the kernel is neither too big nor too small. Le Bars makes use of this delicacy in creating a variant of $\mathcal{K}$, $\mathcal{K}'$, with more predicate letters, such that $\mu(\mathcal{K}')$ does not exist. Thus, the Gödel class without equality fails to obey a 0-1 law.

To involve ourselves any further in Le Bars' proof would necessitate considerable immersion in graph theory, from which he pulls many results and theorems. However, his conclusion interests us insofar as it returns speaks to the connection that was hoped for at the beginning of this paper: the correlation between decidability of a class of first-order sentences and a 0-1 law holding for the associated Σ_1^1 fragment. The proofs that show the failure of the 0-1 laws for the Σ_1^1 fragments that correspond to the other minimal undecidable classes—$\forall\forall\forall\exists$, $\forall\exists\forall$, and $\forall\forall\exists$ with equality—each gave a formula F such that $\mu(F)$ existed, but was not equal to 0 or 1. Le Bars' proof for the Gödel class only gives an F such that $\mu(F)$ does not exist. He admits that he was unable to find an F such that $\mu(F)$ exists and is not equal to 0 or 1. Hence, he conjectures that there may be a rather subtler connection between decidability and 0-1 laws. If we let $\left(\Sigma_1^1(\mathfrak{L})\right)^{AP}$ denote the set of all sentences in $\Sigma_1^1(\mathfrak{L})$ for which the asymptotic probability is defined, then for some fragment $\mathfrak{L}$ of first-order logic,

$$\mathfrak{L} \text{ is decidable if and only if a 0-1 law holds for } \left(\Sigma_1^1(\mathfrak{L})\right)^{AP}$$

If this conjecture is true, it means that decidability is somehow sensitive to a 0-1 law for those sentences in the logic for which the asymptotic probability exists. Moreover, the conjecture points a specific class of sentences that is sensitive to whether or not a logic is decidable. Why does a model-theoretic property—the existence of an asymptotic probability—function as a way to determine if a logic is decidable? What is it about this class of sentences that makes this connection? Provided the conjecture is true, the answers to these questions might give us better insight into what decidability is.

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References

[1] Jean-Marie Le Bars. Fragments of Existential Second-Order Logic without 0-1 laws. In *Logic in Computer Science*, pages 525–536, 1998.

[2] Béla Bollobás. *Random Graphs*. Cambridge University Press, 2nd edition, 2001.

[3] C.C. Chang and H.J. Keisler. *Model Theory*. American Elsevier Publishing Company, Inc., New York, 1973.

[4] Herman Chernoff. A Measure of Asymptotic Efficiency for Tests of a Hypothesis Based on the sum of Observations. *Ann. Math. Stat.*, Vol. 23(No. 4):493–507, December 1952.

[5] Kevin J. Compton. 0-1 Laws in Logic and Combinatorics. *Proc. NATO Advanced Study Institute on Algorithms and Order*, pages 353–383, 1988.

[6] D. Dreben and Warren Goldfarb. *The Decision Problem: Solvable Classes of Quantificational Formulas*. Addison-Wesley, 1979.

[7] Heinz-Dieter Ebbinghaus and Jörg Flum. *Finite Model Theory*. Springer Monographs in Mathematics, 2nd edition edition, 1991.

[8] Ronald Fagin. Probabilities on Finite Models. *Journal of Symbolic Logic*, Vol. 41(No. 1):50–58, 1976.

[9] Y. V. Glebsky, D. I. Kogan, M.I. Liogonky, and V.A. Talanov. Range and Degree of Realizability of Formulas in the Restricted Predicate Calculus. *Cybernetics*, Vol. 5:142–154, 1969.

[10] Warren Goldfarb. The Gödel Class with Equality is Unsolvable. *Bulletin of American Mathematical Soc.*, Vol. 10:180–204, 1984.

[11] Warren Goldfarb. Random Models and Solvable Skolem Classes. *Journal of Symbolic Logic*, Vol. 58(No. 3):908–914, September 1993.

[12] Warren Goldfarb. Informal notes on Ackermann decidability. January 2006.

[13] Yuri Gurevich. Zero-One Laws. *Bulletin of the European Association of Theoretical Computer Science*, February 1991.

[14] Yuri Gurevich and Saharon Shelah. Random Models and the Gödel Case of the Decision Problem. *Journal of Symbolic Logic*, Vol. 48(No. 4):1120–1124, December 1983.

[15] Meredith Jensen. Establishing a Hierarchy for First-Order Quantifier Prefixes. *Harvard College Senior Thesis*, April 2000.

[16] Phokion Kolaitis and Moshe Vardi. 0-1 Laws and Decision Problems for Fragments of Second-Order Logic. *Information and Computation*, Vol. 87:302–338, 1990.

[17] Phokion Kolaitis and Moshe Vardi. 0-1 Laws for Fragments of Second-Order Logic: A Survey. *Proc. of the 25th International Sumposium on Foundations of Computer Science*, 2000.

[18] Phokion G. Kolaitis and Moshe Y. Vardi. 0-1 Laws for Fragments of Second-Order logic: An Overview. *Logic from Computer Science*, pages 265–286, 1991.