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Limits in *Less is Different*[1]

Jeremy Butterfield's lengthy paper *Less is Different* undertakes a defense of a wide range of theses, including claims about reduction, emergence, and supervenience. I want to examine one strand of thought in the paper in which Butterfield offers a justification of the use of limits in scientific idealization. This defense is based on what he calls the 'Straightforward Justification':

Straightforward Justification The use of the infinite limit is justified, despite N being actually finite, by its being *mathematically convenient* and *empirically correct* (up to the required accuracy).

These two notions of mathematical convenience and empirical correctness require definition. The use of an infinite limit is *mathematically convenient* when calculations are more easily performed after taking a limit of some parameter N as $N \longrightarrow \infty$. In these sorts of cases, calculating some value of the system for very large, yet finite, N can be an insurmountable difficulty. This calculation may even have a computational complexity that makes it impossible. However, treating N *as if* it were infinite by means of a limit can make the problem tractable. The use of a limit is *empirically correct* when the error term introduced by the use of the limit can be made as small as desired. More formally, for any $\epsilon > 0$, with ϵ as small as one likes, the error term arising from the use of the limit can be made to be less than ϵ. Most frequently, this ϵ will be determined by the extent of observation error induced by limitations of instrumental measures.

[1] I am grateful to Giovanni Valente for extensive comments and discussions.

I want to raise three issues with Butterfield's project of justifying the use of limits. First, his definition of a limit of a sequence of physical systems is not, in any way, a mathematical limit. Mathematical limits are defined according to strict definitions that require the limit-taking procedure to be conducted on numerical values only. Because systems are not composed of numerical values, neither can limits be taken of them, nor can appeals to mathematical convenience or empirical accuracy be made. This criticism, if convincing, is particularly problematic for Butterfield because the subsequent edifice of his justificatory structure is defined out of sequences of systems and limits thereof.

The second issue I wish to raise examines a distinction he makes between two types of quantities (observables). Essentially, Butterfield adopts a distinction that is perfectly cogent for limits of sequences of numerical values and attempts to apply it to limits of sequences of quantities. My criticism will highlight the importance that sequences of states of systems plays in trying to make sense of a limit of a sequence of quantities, despite the minimal discussion given to the issue by Butterfield. I then conclude with reasons to doubt that definitions of sequences of states of systems are forthcoming easily.

Finally, I focus on Butterfield's attempt to apply the Straightforward Justification in the "Mysterious" case of singular limits, which are (roughly) limits where the infinite limit provides an accurate description of a resolutely finite system. This is of particular importance to Butterfield's thought because he seeks to respond to increased attention paid to singular limits in the literature by demonstrating that there is nothing particularly special at singular limits.[2] I raise objections to Butterfield's toy example, which is meant to be a paradigm of the Straightforward Justification of singular limits. By doing so, I hope to show that the formal definition of singular

[2]This motivation is particularly evident in his strong disagreement with Batterman [1, 2, 3, 4] and Rueger [8, 9]. In so demonstrating that nothing much is special about singular limits, he is roughly aligned with Belot [5] and Hooker [7].

differs from the way Butterfield actually uses the term. His actual use relies heavily on his definition of limits of sequences of systems and his distinction between two types of quantities—the very things objected to in the previous two sections.

§1 will discuss the strict definition of mathematical limits for functions and sequences and draw three morals. §2 will introduce the formal terminology of Butterfield's approach and object that a limit of a sequence of states is ill defined. §3 will introduce the distinction between the two types of values for quantities and highlight its dependence on sequences of states of systems, which I will cast doubt upon. Finally, §4 will introduce the Mystery of Singular Limits and raise objections to Butterfield's toy example.

1 Mathematical Limits

There are two types of limits used in mathematics: limits of functions and limits of sequences.

1.1 Limits of Functions

Limits of functions provide a formal statement of the informal notion that the value of a function can be made arbitrarily close to some number, given that the input variable is made "close enough" (but not equal) to some other value. That is,

$$\lim_{x \longrightarrow c} f(x) = L \tag{1}$$

can be read informally as saying that the value of the function $f(x)$ can be made arbitrarily close to L by making x sufficiently close to c. Now, the terms "arbitrarily close" and "sufficiently close" are imprecise and informal. They are rigorized by the introduction of the $\delta - \epsilon$ definition of limits.

Definition 1. *Let f be a function that is defined on an open interval that contains c, except maybe at c. Let L be a real number. The equation*

$$\lim_{x \longrightarrow c} f(x) = L$$

means that for every $\epsilon > 0$, there is some $\delta > 0$ such that

$$0 < |c - x| < \delta \Longrightarrow |L - f(x)| < \epsilon.$$

This definition can also be expressed purely symbolically as follows. For domain D:

$$(\forall \epsilon > 0)(\exists \delta > 0)(\forall x \in D)\left((0 < |c - x| < \delta) \Longrightarrow (|L - f(x)| < \epsilon)\right).$$

Note first that limits can be taken even if the function is not defined at c. For instance, the function

$$g(x) = \frac{x^2 + 1}{x - 1}$$

is not defined at $x = 1$, but the limit as x approaches 1 *is* defined. It is $\lim_{x \longrightarrow 1} g(x) = 2$.[3]

Secondly, note that we can also take a limit if $c = \infty$. Instead of being defined as within δ of c, a limit as x tends towards infinity is defined in terms of being bigger than some δ. So,

$$\lim_{x \longrightarrow \infty} f(x) = L$$

means that for every $\epsilon > 0$, there exists a $\delta > 0$ such that

[3] A function is said to be *continuous* at a point when its limit and its value at that point agree. $g(x)$ is *discontinuous* at 1 because $\lim_{x \longrightarrow 1} g(x) = 2$ but $g(1)$ does not exist. The function is, however, continuous at all other points.

$$(x > \delta) \Longrightarrow |f(x) - L| < \epsilon.$$

Again, a purely symbolic representation is:

$$(\forall \epsilon > 0)(\exists \delta > 0)(\forall x \in D)\left(x > \delta \Longrightarrow |L - f(x)| < \epsilon\right).$$

Third, and finally, note that a limit *diverges* when $L = \infty$. This means that for any $n \in \mathbb{R}$ (no matter how big), if we make x sufficiently close to c, the value of the function at x will be greater than n. Symbolically,

$$\lim_{x \longrightarrow c} f(x) = \infty$$

means

$$(\forall n \in \mathbb{R})(\exists \delta > 0)\left((|x - c| < \delta) Longrightarrow (f(x) > n)\right).$$

To say a limit diverges is to say that the function can be made arbitrarily large by taking the value of a function for x sufficiently close to c.

1.2 Limits on Sequences

Limits can also be taken of sequences. Consider a sequence of real numbers, with $n \in \mathbb{N}$:

$$a_1, a_2, \ldots, a_n, \ldots$$

Taking a limit of this sequence consists in letting n grow without bound towards infinity. Thus, a limit of a sequence is defined.

$$\lim_{n \longrightarrow \infty} a_n = L$$

means that for any $\epsilon > 0$, there exists some $N \in \mathbb{N}$ such that

$$(n > N) \Longrightarrow (|a_n - L| < \epsilon)\,.$$

The relationship between limits of functions and limits on sequences should be apparent. Like limits of functions, limits of sequences can fail to exist. Additionally, a limit on sequences is said to diverge when $L = \infty$.

1.3 Three Morals

There are three important morals about limits to bear in mind. The first is that both of these types of limits deal with numerical values. The reason for this restriction is that the distance metric (in these cases the absolute value function) has to be defined such that the notions of ϵ and δ closeness make sense. If there is a discrete metric under which the choice of δ is restricted, then there are possible situations where an appropriate δ cannot be chosen.

For functions, the definition of the limit is based on choosing δ "close enough" to c for ϵ close to L. Thus, to guarantee that for a given ϵ the range of the function has values within the ϵ-ball around L, the domain of the function must be sufficiently dense to ensure that there are xs within the δ-ball around c. If the domain of the function is not sufficiently dense, then there could be a δ-ball around c that contains no xs for which the function is defined.

For sequences, the situation is slightly different. When taking the limit of a sequence, the ns can only take on natural number values; there are no δ-balls to be concerned with. There is, therefore, no restriction on the density of the values taken on by the a_i. However, what are required are well-defined notions of *distance from* (when $L \neq \infty$) and *greater than* (when $L = \infty$), which require a metric and a well-ordering, respectively. Although it is possible to define a perverse metric and ordering, the applications under consideration mean that we need only consider real numbers. Thus, the first moral to take from the definition of limits is that **limits deal with numbers**.

The second moral to take away from the definition of limits is that there is a distinction between the value *at*-the-limit and the value *of*-the-limit for limits of functions. The value of-the-limit is L in what was defined above. However, the value at-the-limit is the value of the function obtained by letting $x = c$, that is, the value at-the-limit is $f(c)$.

These two values—L and $f(c)$—need not 1) be defined, nor 2) agree with each other. We saw an example of the first with $g(x) = \frac{x^2+1}{x-1}$. $g(1)$ is undefined, but $\lim_{x \longrightarrow 1} g(x) = 2$. An example of the second can be created by defining a piecewise function $g_1(x)$ defined as follows:

$$g_1(x) := \begin{cases} g(x); if x \neq 1 \\ 0; if x = 1 \end{cases}$$

For $g_1(x)$, the value of-the-limit and at-the-limit both exist, but they disagree $(2 \neq 0)$. The second moral is that **for limits on functions, there is a distinction between at-the-limit values and of-the-limit values** whenever $c \in \mathbb{R}$.

The third moral applies to limits on sequences. For sequences, a limit can only be taken when $n \longrightarrow \infty$ (the limit as $x \longrightarrow N$ does not exist because the limits from the left and right are a_{N-1} and a_{N+1}, respectively, and they do not agree). Because of this restriction, whether there is a difference between the value of-the-limit and the value at-the-limit for sequences is in one sense trivial, and in another sense, quite difficult.

It is trivial because the value at-the-limit is not well-defined. Recall that n, the subscript of the sequence, was a natural number. ∞ is not a natural number, so, strictly speaking, a_∞ is not well-defined and does not make sense. Whether there is a difference between the value of-the-limit and the value at-the-limit is trivial because the second is not defined, so there can be no "difference".

The question is tricky, however, because there can be natural extensions of the sequence to ∞ and beyond. If we consider the sequence defined by $a_n := n^2$, then it is possible to extend the sequence for $n = \infty = \omega$. The question whether the value of-the-limit equals at-the-limit can then be meaningfully asked, but only on that extension because the calculation of a_∞ is defined for that extension. The third moral is that **the at-the-limit/of-the-limit distinction for sequences is only possible when a_∞ is independently well-defined**. By 'independently', I mean that for the distinction to be possible in principle, the definition of a_∞ cannot be mere shorthand for $\lim_{n\longrightarrow\infty} a_n$.

Each of these three morals will play an important role in the following criticisms. The first will be featured in §2, while the second and third will be involved in §3 and §4.

2 When a Limit is not a Limit

2.1 Systems, Quantities, Values

In his paper, Butterfield focuses on three elements: systems, quantities, and values.

System A system is some physical system of interest denoted by σ. These systems can take many forms. In statistical mechanics, a physical system can be something as simple as a cup of coffee or a box of gas composed of some number of particles.

In probability theory—an example given by Butterfield himself—a system can be a roulette wheel divided into alternating red and black wedges.

These σ-systems are defined in terms of a parameter $N \in \mathbb{N}$, which determines some feature of the system, e.g., the number of particles of the gas in the box. Strictly speaking, 'σ' on its own is bad notation because it denotes a class of systems rather than an individual system. σ indicates the type of physical system we are talking about—e.g., coffee cup or box of gas—while the choice of N selects a particular system of that type.

Quantity Each type of system has observable *quantities* by which we describe features of that type of system. Different types of physical systems have different quantities. Examples of quantities include, but are not limited to, momentum, position, temperature, energy, or spin. In the case of gas in a box, the relevant quantities might include momentum or position.

Importantly for Butterfield's conception of quantity, quantities are only defined over types of systems and not independently of those particular systems. Each quantity is limited to a particular type of system. Thus, a quantity f is defined over a system $\sigma(N)$ as $f(\sigma(N))$.

Value Finally, a quantity of a system only obtains a numerical *value* when the physical system takes on some *state*. A state of a system is some arrangement of the physical stuff that comprises the system; the state is what determines the numerical values of the quantities on that system. It is not until the state of the system, denoted s_N, is specified that the value of the quantities can be determined. Additionally, the state of the system characterizes the system. This dependence on states marks an important distinction between quantities and values: quantities make no reference to states while values depend on states. This point will become particularly important later when trying to define a limit for sequences of values.

We can only talk about a numerical value when a i) particular physical system, ii) quantity, and iii) state have been specified. Accordingly, we can think of a quality defined on a system as a function from states to numerical values, each of which depends on the type of system (σ) and the parameter N. Notationally, a value is represented as $v\,(f(\sigma(N)), s_N)$. However, Butterfield suppresses the involvement of states in his notation and writes values as $v(f(\sigma(N)))$.

For the reader's convenience, the notation used by Butterfield for systems, quantities, and values is as follows.

- $\sigma(N)$ is the *system* defined by some parameter N.
- $f(N) := f(\sigma(N))$ is some *quantity* on the system $\sigma(N)$.
- $v(f(N)) := v(f(\sigma(N)))$ is the *value* of some quantity on the system $\sigma(N)$.

2.2 Limits of Sequences of Systems, Quantities, and Values

The definition of a system, dependent as it is on a parameter $N \in \mathbb{N}$, suggests the notion of a sequence of systems.

$$\sigma(1), \sigma(2), \ldots, \sigma(N), \ldots \qquad \text{(System Sequence)}$$

Note also that this sequence of systems also gives rise to a sequence of quantities defined over those systems:

$$f(1), f(2), \ldots, f(N), \ldots \qquad \text{(Quantity Sequence)}$$

Finally, assuming that there is an associated sequence of states (s_N, which is suppressed in Butterfield's notation), there is a sequence of numerical values of quantities:

$$v(f(1)), v(f(2)), \ldots, v(f(N)), \ldots \qquad \text{(Value Sequence)}$$

These sequences suggest the use of mathematical limits. Butterfield takes there to be three questions regarding such uses. He writes,

> (1) One can ask whether this sequence [of systems] has as a limit, in the sense of there being (as a mathematical entity), a natural well-defined infinite system $\sigma(\infty)$.
>
> (2) One can ask whether a sequence of quantities on successive systems, say $f(N) := f(\sigma(N))$, has a limit, which we might denote by $f(\infty)$ (Of course, the physical idea of each member of such a sequence will be in common, e.g. energy or momentum: but we distinguish the members by their being quantities on different (sizes of) system.)
>
> (3) Finally, one can ask whether a sequence of real number values on quantities on successive systems, say $v(f(N)) := v(f(\sigma(N)))$, has a limit.[4]

These questions are about the existence of limits of sequences of systems, quantities, and values of quantities, respectively. There are two senses in which Butterfield is asking the question. The first question is whether sense can be made of the notion of taking the limit of systems, quantities, and values of quantities, respectively. The question is whether, for instance, a limit of a sequence of systems can be well-defined. The second sense of the question is: given that the notion of taking the limit is well-defined for sequences of systems/quantities/values of quantities, what is that limit. In terms of our previous notation, answering the first sense of question requires providing a definition of a limit, while answering the second sense requires a determination of some system/quantity/value of a quantity. The first sense of the question is about the

[4] [6, p. 12]. A particularly puzzling aspect of Butterfield's characterization of the limit of a sequence of systems is his declaration that the infinite system is a "mathematical entity." Systems are meant to be physical structures that are realized in the world, not merely mathematical models. It is not hard to imagine his motivation, given that he wants to deny (emphatically) that infinite systems are realized. But, if that is the case, he needs to explain how a sequence of physical systems has a mathematical entity as its limit.

generic definition of limits; the second sense of the question is about some *particular* limit.

In the case of a sequence of values of quantities, the answer to the first sense of the question is affirmative, since the standard definition of limits applies. A sequence of values is merely a sequence of real numbers, and we have seen above how such a limit is defined. For systems and quantities, however, the given mathematical definition of limits is unacceptable because of our first moral: limits can only be taken of numerical values. Applying a limit to sequences of systems and values requires extending the ordinary definition of limit-taking beyond its numerical basis.

Butterfield takes there to be a viable candidate for such an extension of the definition of limits to systems and quantities. He writes,

> But we can also make sense of the first two questions. As to (1): in both classical and quantum physics we can often define the limit of a sequence $\sigma(N)$. Some approaches individuate a system by its state-space, and then use infinite cartesian or tensor products (for the classical and quantum cases respectively). Other approaches individuate a system by its set (in fact: algebra) of quantities, and then define limit algebras. This leads to how we make sense of (2). The algebra of quantities usually has a mathematical structure (in particular a topology) that enables one to define the limit of a sequence of quantities (i.e. not just, as in (1), a limit of a sequence of their values).[5]

This is, unfortunately, all we get regarding how to extend the definition of limit to systems or quantities. I want to focus on how Butterfield defines the limit of a sequence, leaving aside for the moment the question of how to define the limit of a sequence of quantities. I will take up the issue of defining limits of sequences of quantities in the next section.

Taking the case of classical physics as an example, the picture is presumably that a system is described by a $6N$-dimensional Cartesian space, with the six dimensions representing the x, y, and z coordinates of position and momentum, respectively.

[5] [6, p. 13].

This N is the parameter that we let approach ∞, and thus, the limit Cartesian space has infinite dimension.

2.3 The Problem with the "Limit" of a Sequence of Systems

Two things about this approach to defining limits of sequences are important. First, the sequence of systems is *not* a sequence of numerical values. A system is a physical system such as a box of gas or a cup of coffee. From the first moral above, it is the case that limits, as defined strictly, only deal with numerical values that can be accommodated in an ϵ-definition. So, any attempt to take a limit of a sequence of systems would be a complete redefinition of the limit from the strict definition above.

Second, this redefinition defines taking the limit of the sequence $\sigma(N)$ as the system resulting from setting the parameter $N = \infty$. Symbolically, this approach defines the limit in the following way.

$$\lim_{N \longrightarrow \infty} \sigma(N) := \sigma(\infty).$$

The at-the-limit/of-the-limit distinction for sequences of systems appears to be, in principle, impossible because the of-the-limit system is *defined as* the system at-the-limit. This limit-taking is, in fact, not limiting-taking behavior at all because there is no mathematical definition of the limit. Instead, "taking the limit" of a sequence of systems is determining what the system would look like if $N = \infty$.

This determination of the system when a parameter is infinite runs afoul of the same problem that prompted our third moral above: taking the limit only makes sense when the case where $N = \infty$ is well-defined. However, one of the main applications of limit taking is for systems where physical systems with $N = \infty$ is not a well-defined system. Given that the cases under examination are precisely those where $N = \infty$ is *not* a well-defined notion for the physical systems (e.g., the number of particles in a system cannot be infinite), defining the limit of a sequence of systems as equal to the system at-the-limit presents the first major problem with Butterfield's analysis. He sets out to justify the use of limits in scientific idealizations, but by defining the "taking of a limit" of a sequence of systems as considering $N = \infty$, there is no more taking of limits to be justified for systems.

3 Limits of Sequences of Quantities

Recall that Butterfield's project seeks to justify the use of infinite limits in scientific idealization using the straightforward justification of mathematical convenience and empirical accuracy. However, in order to apply this justification to *all* instances of mathematical limit taking, Butterfield must respect a distinction I drew earlier for limits on functions between values of-the-limit and values at-the-limit. This distinction allows Butterfield to address the "mystery" of *singular* limits, which are defined in the following way.

Singular Limit: An infinite limit is singular when

1. The value of-the-limit exists;
2. The value at-the-limit exists;
3. They do not agree;
4. The value at-the-limit is empirically accurate.

In the current section, I will to detail exactly how Butterfield conceives the at-the-limit/of-the-limit distinction and argue his application of the distinction to limits of sequences of quantities depends crucially on a notion that he suppresses: sequences of states of systems. I will close the section with reasons to doubt that sequences of states are simple enough to be suppressed. §3.1 is of continued importance in §4, for it is through this distinction that singular limits are defined.

3.1 Drawing the Distinction

Butterfield draws the of-the-limit/at-the-limit distinction for both quantities and values. He starts with two suppositions. Suppose

> a) a sequence $v(f(N))$ of values of a quantity has a limit $\lim_{N\longrightarrow\infty} v(f(N))$ as N tends to infinity (as mentioned in Section 3.1, a sequence of states s_N is here understood, so that one might write $v(f(N), s_N)$). And
>
> b) there is also a well defined infinite system $\sigma(\infty)$ on which: b.1)] the common physical idea of the various $f(N)$ makes sense and gives a natural well- defined limit quantity, which we might write as $f(\sigma(\infty))$ (on $\sigma(\infty)$); and b.2)] on which there is a natural well-defined limit state, s say.
>
> Then we need to distinguish:
>
> (i) the given limit of the values, $\lim_{N\longrightarrow\infty} v(f(N)) \equiv \lim_{N\longrightarrow\infty} v(f(N, s_N))$ from
>
> (ii) the value $v(f(\sigma(\infty), s))$ of the natural limit quantity $f(\sigma(\infty))$ in the natural limit state, s.[6]

In the quotation, (i) is the value of-the-limit and (ii) is the value at-the-limit. To reiterate from the earlier discussion of limits on sequences, the of-the-limit value is the result of applying a calculus limit to a sequence of values changing across a single parameter. The at-the-limit value is the result of substituting ∞ for the parameter and calculating the resultant value. The at-the-limit value only makes sense given that there "is a well-defined infinite system $\sigma(\infty)$." The at-the-limit value of the sequence is the $(\omega + 1)$th element in the sequence since we are taking the limit as $N \longrightarrow \infty = \omega$.

The distinction can be illustrated most clearly by considering a sequence of sets (systems) that contain the natural numbers:

[6] [6, p. 14].

$$\sigma(1) = \{1\}$$
$$\sigma(2) = \{1, 2\}$$
$$\vdots$$

Let $f(\sigma(N))$ be the ordinal type function, $ord(\sigma(N))$. The values of the function are as follows:

$$\begin{aligned}
v(f(\sigma(1))) &:= v(f(1)) = 1 \\
v(f(\sigma(2))) &:= v(f(2)) = 2 \\
&\vdots \\
v(f(\sigma(i))) &:= v(f(i)) = i \\
&\vdots
\end{aligned}$$

This sequence of values increases as N increases. We can also take a limit of this sequence of values,

$$\lim_{N \longrightarrow \infty} v(f(N)) = \omega$$

This limit has the form of the ordinary calculus limit—it is the result of determining the value of a numerical quantity obtained by letting a single parameter get increasingly large towards ∞. The limit diverges because for any $M \in \mathbb{N}$, there exists an $N \in \mathbb{N}$ such that $v(f(N)) > M$. Note that for our example, the limit is equal to the ordinality of the natural numbers, which makes sense given that taking the limit in some way "runs through" all of the natural numbers and "arrives" at the set containing all of the natural numbers.

However, the value of-the-limit is not the same as the value at-the-limit. In this case, the at-the-limit system is obtained by letting $N = \omega$. But, $v(f(\omega)) \neq \omega$ because $\sigma(\omega)$ does not have order ω. Recall that for every i,

$$\sigma(i) := \{1, 2, \ldots, i\}$$

Thus,

$$\sigma(\omega) := \{1, 2, \ldots, \omega\}$$

which has order $\omega + 1$. Thus, $v(f(\omega)) = \omega + 1$. In this example, there is a numerical difference between the value of-the-limit (ω) and the value at-the-limit ($\omega + 1$). This difference is only possible when $N \in \mathbb{N} \cup \{\omega\}$.

3.2 Distinction for Quantities

Butterfield wants to draw this same distinction for quantities that he does for values. In fact, he takes it that the distinction is identical, save for the involvement of states of the system. He writes,

> For quantities themselves, rather than values, the point is in essence the same. The statement is a close parallel of that in (a): indeed, shorter since we refer only to quantities, not to values of quantities—albeit thereby more abstract. Thus suppose: (i) a sequence of quantities $f(N)$ has a limit, dubbed $f(\infty)$ in Section 3.1. And suppose also: (ii) there is also a well-defined infinite system $\sigma(\infty)$ on which the common physical idea of the various $f(N)$ makes sense and gives a natural well-defined limit quantity, which we might write as $f(\sigma(\infty))$ (on $\sigma(\infty)$). Then we need to distinguish:
>
> (i) the given limit, $f(\infty) := \lim_{N \longrightarrow \infty} f(N)$, from
>
> (ii) the natural definition of the quantity $f(\sigma(\infty))$ on $\sigma(\infty)$.[7]

The distinction that Butterfield means to draw, then, is between the quantity on the infinite system and the limit of the sequence of quantities. Because of the definition

[7] [6, p. 12].

of $f(N)$ and the fact that $\sigma(\infty) = \lim_{N \longrightarrow \infty} \sigma(N)$[8], the distinction drawn is between $\lim_{N \longrightarrow \infty} f(\sigma(N))$ and $f(\lim_{N \longrightarrow \infty} \sigma(N))$. The claim is that in principle, these two quantities can be different. They need not be different, but they *can* be.

Given the suppositions of the distinction as laid out, the definition of the quantity on the infinite system is well-defined. Therefore, $f(\lim_{N \longrightarrow \infty} \sigma(N))$ is a well-defined quantity. The question then becomes how to make sense of $\lim_{N \longrightarrow \infty} f(\sigma(N))$. Recall first the strict definition of limits, which deal only with numerical values. For a sequence to have a limit L, for any arbitrary $\epsilon > 0$, there must be some element in the sequence for which the distance between L and any later element of the sequence is less than ϵ. However, in the case of sequences of quantities, no such convergence (or divergence) is possible, since quantities are *not* numerical values. The $\lim_{N \longrightarrow \infty} f(\sigma(N))$ cannot be defined in terms of limits on numerical sequences.

It might, however, be defined as a limit of a sequence of functions. Pointwise convergence of a sequence of functions is given by the following general definition. Given a domain D,

$$\lim_{N \longrightarrow \infty} f_N(x) = g(x) \Longleftrightarrow (\forall x \in D) \left(\lim_{N \longrightarrow \infty} f_N(x) = g(x) \right)$$

In words, this definition means that a sequence of functions f_N approaches (converges to) a function g(x) if for every $x \in D$, every sequence of values of $f_N(x)$ converges to the value of $g(x)$. In other words, to prove pointwise convergence of $f_N(x)$ to $g(x)$, one must show that for some $\epsilon > 0$ and every x, there is some $m \in \mathbb{N}$ such that $|f_m(x) - g(x)| < \epsilon$ for every $N > m$.

There are a few important things to take from the definition of pointwise convergence. First, it is built up out of limits of sequences of values; the left hand side of the definition refers only to limits of functions, while the right hand side only refers

[8]By definition, see p. 13.

to limits of sequences of values of functions. Second, note that the domain of the functions $f_N(x)$ and $g(x)$ is *identical* and *fixed*. Both $f_N(x)$ and $g(x)$ go from D to numerical values.

Therefore, in order to apply the pointwise definition of limits to quantities, we must think of quantities as functions. But, in order to treat them as such, we must first determine their domain and range. The latter is the easier of the two, for it is merely numerical values, be they real, natural, rational, or what have you, depending on circumstance and the physical interpretation of the quantity. In general, the domain of the quantities when considered as functions consists of states of systems.

It is also important to clarify the role of the parameter N in the reimagining of quantities as functions. N determines the dimension of the states of the system, and therefore, N determines the dimension of the state space. In this way, N also determines the shape of the functions; a state with N-many dimensions will give rise to a function that takes N-many inputs.

With these clarifications in hand, we can address how Butterfield seeks to apply pointwise convergence of sequence of functions to sequences of quantities. Unfortunately, the notation used by Butterfield obscures the point I am trying to make, so for the purposes of this definition, I will take quantities in the following way.

$$f(\sigma(N))(\bar{x}) : S_N \longmapsto \mathbb{R}$$

is a function defined for a system $\sigma(N)$ that goes from the state space of the system (S_N) to $\mathbb{R}$ We can then apply pointwise convergence in the following way, bearing in mind that $\bar{x}$ in this context refers to a state of the system, which can be of any finite dimension. Given S_N, which is the state space with dimension determined by N,

$$\lim_{N \longrightarrow \infty} f(\sigma(N))(\bar{x}) = g(\sigma(\infty))(\bar{x}) \Longleftrightarrow$$
$$(\forall \bar{x} \in S_\infty)\left(\lim_{N \longrightarrow \infty} v(f(\sigma(N))(\bar{x}) = v(g(\sigma(\infty)))(\bar{x})\right)$$

This definition of pointwise convergence of one quantity to another quantity is a direct analogue of the definition of pointwise convergence above. Again, the left hand side of the definition deals entirely with quantities (functions), while the right hand side of the definition only refers to values of quantities (functions). Additionally, the definition, as before, quantifies over all elements in the domain on the right hand side. So, the definition of pointwise convergence of quantities is built up out of the convergence of values of that quantity for every state in the state space.

However, in defining convergence of quantities in this way, a problem arises. In the standard definition of pointwise convergence, the domain is fixed. However, in the definition of pointwise convergence for quantities, two things are converging: the values of the functions and the domains of the functions. Each quantity $f(\sigma(N))(\bar{x})$ is only defined over state spaces that have N-many dimensions. That is, $f(\sigma(N)(\bar{x})$ is a function from state spaces of dimension N to the real numbers; it is, for instance, not defined over state spaces that have infinitely many dimensions, i.e. for S_∞. So, in order to have a change at being well-defined, there must be a way to make the sequence of states of the system "line-up" appropriately with the sequence of quantities that we are taking the limit of.

Unfortunately, the notion of taking limits of states of systems is treated exceedingly briefly by Butterfield. The entirety of his comments on the matter is as follows:

> [Regarding limits of values] Here a sequence of states, s_N say, on the $\sigma(N)$ is to be implicitly understood, so as to define values for the quantities $f(N)$; but to simplify notation, I will for the most part not mention s_N,

and indeed take states as understood.[9]

In the definition of a sequence of values, the states can be suppressed, since they are input variables and all we care about in that context is the output values. However, the definition of a sequence of quantities must quantify over all possible states in the infinite state space, so Butterfield cannot rely on a single *fixed* sequence of states. There must be some account of taking limits of sequences of states of systems.

As an example of the complexity involved in taking limits of sequences of states, consider a greatly simplified example. Let $\sigma(N)$ be a system with N elements that can take the value of 1 or 0. Let s_∞ be a state of this infinite system given by the following sequence of values for elements: $\{\alpha_1 = 1, \alpha_2 = 0, \alpha_3 = 1, \alpha_4 = 0, \ldots\}$. Suppose further that the value under consideration is the numerical average. of all the terms in the sequence. Then, there are two different sequences of states that converge to a_∞ (in one sense), but give vastly different values. Consider the following two sequences:

$$\text{Sequence 1}: \{1\}, \{1,0\}.\{1,0,1\}, \{1,0,1,0\}, \{1,0,1,0,1\}, \{1,0,1,0,1,0\} \ldots$$
$$\text{Sequence 2}: \{1\}, \{1,1\}.\{1,1,0\}, \{1,1,0,1\}, \{1,1,0,1,1\}, \{1,1,0,1,1,0\} \ldots$$

These two sequences both "converge" to a_∞ in the sense that both sequences can have their 1s and 0s put into one-to-one correspondence with the respective 1s and 0s in a_∞. However, if we take the limit of the numerical average of the terms contained in Sequence 1 and Sequence 2, we find that the limit of the averages are $\frac{1}{2}$ and $\frac{2}{3}$, respectively. So in one sense of "convergence", these two sequences are the same, but their values differ for a choice of quantities.

[9] [6, p. 12].

Drawing the distinction between of-the-limit quantities and at-the-limit quantities, then, rests crucially on assumptions that underlie taking the limit of a sequence of states, at least if we are to define it in terms of a limit of a sequence of functions, which seems to be the only possible definition. This underlying framework is absent from Butterfield's account. For the time being, I will put to the side the issues underlying limits on quantities and focus solely on limits of values, a concept that, for the moment, does not have any problems.

4 Singular Limits and Toy Examples

4.1 The Mystery of Singular Limits

Recall that Butterfield seeks to apply the Straightforward Justification of mathematical convenience and empirical accuracy to the use of mathematical limits in scientific idealizations. The distinction between values of-the-limit and values at-the-limit gives rise to a four-part classification of limits.

(I) The value of-the-limit exists and the value at-the-limit does not exist;

(II) The value of-the-limit does not exist and the value at-the-limit exists;

(III) The value of-the-limit exists and the value at-the-limit exists and they agree;

(IV) The value of-the-limit exists and the value at-the-limit exists and they disagree.

The appeal to the Straightforward Justification in Cases I-III is immediately expected, according to Butterfield. For cases I and II, there is only one value to consider, so that the value is empirically accurate is sufficient to justify the use of the limit. Similarly, case III presents only one value because the at-the-limit and of-the-limit values are equal.

Only Case IV presents a problem, and even then according to Butterfield, only when the value at-the-limit is the empirically accurate value. The reason that this constitutes a mystery is that the systems for which the values of the quantities are defined are strictly finite systems. If the value at-the-limit—the value of the quantity as defined on the infinite system $\sigma(\infty)$—is empirically accurate despite the existence of a value of-the-limit, then that accuracy seems to imply that the physical system has infinitely many degrees of freedom. Returning again to the box with N-many gas molecules, a singular limit would imply that the physical system has an infinite number of molecules, a result that directly contradicts the finitude of matter. Indeed, all the physical systems under consideration have only a finite number of degrees of freedom. Thus, the Mystery: how can Butterfield Straightforwardly Justify empirically accurate uses of values at-the-limit when their physical interpretation violates basic physical principles?

4.2 Dissolving the Mystery of Singular Limits

The mystery of singular limits, according to Butterfield, can be dissolved by appealing to values of other functions, which achieve empirical accuracy long before the infinite limit. The mathematical convenience of the Straightforward Justification attaches to the infinite limit, while the empirical accuracy attaches to both the infinite limit and this separate, finite cousin-function. The calculation using the value at-the-limit is merely a convenient way of approximately (within ϵ) the actual value of the finite cousin function.

The picture is clarified through symbolic representation. Let $f(\sigma(N))$ be some quantity on a physical system $\sigma(N)$, and let the limit of the sequence of the values be singular. That is, let

$$\lim_{N \longrightarrow \infty} v(f(\sigma(N))) \neq v(f(\sigma(\infty))),$$

and let $v(f(\sigma(\infty)))$ be empirically accurate. Butterfield's claim then is that for every such quantity $f(N)$, there is some other quantity $g(N)$ such that for some N, $v(g(\sigma(N)))$ is an empirically accurate value for $f(\sigma(N))$. Butterfield writes,

> [In singular limits,] there are other quantities, for which (despite f's singular limit) the finite-N model, for large N, is close to the values given by the infinite model and is thereby also empirically correct. In fact, these other quantities are 'cousins' of the quantity f which we first considered.[10]

He further illustrates this method of cousin functions using a toy mathematical example. Consider a sequence of real functions:

$$g_N(x) = \begin{cases} -1 : & x \leq -\frac{1}{N} \\ Nx : & -\frac{1}{N} \leq x \leq \frac{1}{N} \\ 1 : & \frac{1}{N} \leq x \end{cases}$$

Butterfield takes it that "the sequence has as its limit the function g_∞ given by:"[11]

$$g_\infty(x) = \begin{cases} - \quad 1 : x < 0 \\ \quad 0 : x = 0 \\ \quad 1 : x > 0 \end{cases}$$

The sequence of g_Ns is meant to exemplify the mystery of singular limits along the following lines. Every g_N with finite N is a continuous function, but g_∞ is discontinuous, making the limit 'singular' because "continuity is lost."[12] Further, suppose that calculating g_N for very large N is difficult while using g_∞ is mathematically convenient. All that is left to demonstrate the mystery of singular limits is empirical

[10] [6, p. 16].
[11] [6, p. 17].
[12] [6, p. 17].

accuracy, which Butterfield takes it, is guaranteed for large enough N. Then, if N is large enough, the shape of the graph of values of the function will be sufficiently close to $g_\infty(x)$ that it is empirically indistinguishable. That is, given a physical interpretation of $g_N(x)$ and $g_\infty(x)$, no measurement could differentiate between $g_N(x)$ and $g_\infty(x)$.

4.3 Problems with the Toy Example

However, the problem with the above reasoning is that the limit of the sequence of g_Ns is not singular by *Butterfield's own lights.* Recall that part of being singular is a disagreement between the values of-the-limit and at-the-limit. However, I claim that Butterfield has explicitly defined the limit of the sequence of g_Ns as g_∞. He writes, "This sequence has, as its limit, the function g_∞, defined by ..."[13] This "limit-taking" behavior is not defined over the whole domain of the functions because g_∞ is discontinuous at $x = 0$. The sequence of g_Ns do not converge pointwise to g_∞ precisely because of that discontinuity. Put another way, consider a different infinite function g'_∞, which is identical to g_∞ except $g'_\infty(0) = 1$. This new function agrees with the limit of $g_N(x)$ to the same extent that g_∞ does. There is no difference, from the point of view of taking a limit of the sequence of g_Ns, between g_∞ and g'_∞.[14] Thus, Butterfield is not taking a limit in referring to g_∞, but rather explicitly defining $\lim_{N\longrightarrow\infty} g_N(x) = g_\infty(x)$, which means that this limit cannot be singular because there is no disagreement between values of-the-limit and at-the-limit.

In order to gin up the needed disagreement, Butterfield introduces a two-valued quantity f_N such that,

> f_N, $N \in N \cup \{\infty\}$ that encodes whether or not g_N is continuous: $f_N := 0$

[13] [6, p. 17].

[14] There are good practical reasons to prefer g_∞ over other functions, for instance the fact that $g_N(0) = 0$ for all N. But this practical consideration does not make the sequence of functions converge to g_∞ in any mathematical sense.

if g_N is continuous and $f_N := 1$ if g_N is discontinuous. Then we have: $f_N = 0$ for all finite $N \in N$, but $f_\infty = 1$."[15]

Because $f_N = 0$ for all $N \in \mathbb{N}$, $\lim_{N \longrightarrow \infty} f_N = 0$. However, $f_\infty = 1$. Taking the limit of the f_Ns provides genuine disagreement between the values of-the-limit and the values at-the-limit.

But a counterpart of the previous problem arises for the f_Ns when considering whether the limit is singular or not. Although there is genuine disagreement, it is not at all clear that the f_∞ is empirically accurate while $\lim_{N \longrightarrow \infty} f_N$ is not. If the function under question is some g_N with finite N, then the empirically accurate f-function is f_N, for every g_N is continuous. It would be empirically inaccurate to use f_∞ to describe the continuity of any g_N, with finite N. Indeed, the only empirical accuracy appealed to by Butterfield is the empirical accuracy of g_∞.

Even in this simple, toy mathematical example, a number of issues arise. First, taking the limit of the sequence of functions does not automatically generate an infinite function for all potentially relevant values. Only by defining the infinite function in a particular way does the infinite function gain its empirical accuracy by which it is meant to be straightforwardly justified. But the Straightforward Justification is meant to apply to the use of limits, not merely the definition of infinite functions, and in this case, limit taking is not taking place. Second, even though the toy example is meant to illustrate how to resolve the mystery of singular limits, it is not at all clear that we have a singular limit. The limit of the sequence of the g_Ns (apart from $x = 0$) agrees with g_∞, so there is no disagreement between the value of-the-limit and the value at-the-limit. The limit of the sequence of f_Ns and f_∞ disagree, but it is not clear what f_∞ is meant to be an empirically accurate description of.

[15] [6, p. 17].

5 Conclusion

The problems of the toy example generalize. The main problem stems from a discrepancy between how Butterfield defines singular limits formally, and the way he uses the term in his examples. His use of ‘singular’ indicates that the value at-the-limit exhibits some feature that does not appear for any finite value. More precisely, it is not the *value* that exhibits this new feature, but the infinite system or quantity. This point can be witnessed in the toy example, for Butterfield appeals to a feature about the quantity (function)—its continuity or lack thereof—to demonstrate that the limit is ‘singular’. Singularity, as actually used by Butterfield, is a claim about a unique feature of the at-the-limit quantity or system. But I hope that §2 and §3 raised issues for Butterfield’s definitions of limits of sequences of systems and quantities.

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